C^1- APPROXIMATE SOLUTIONS OF SECOND ORDER SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work a new method is developed to obtain C^1-approximate solutions of initial and boundary value problems generated from a one-parameter second order singular ordinary differential equation. Information about the order of approximation is also given by introducing the so called *growth index* of a function. Conditions are given for the existence of such approximations for initial and boundary value problems of several kinds. Examples associated with the corresponding graphs of the approximate solutions, for some values of the parameter, are also given.

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1. Introduction

A one-parameter perturbation singular problem associated with a second order ordinary differential equation is a problem whose the solutions behave nonuniformly near the initial (or the boundary) values, as the parameter approaches extreme levels. In this work we develop a new method to obtain approximate solutions of some problems of this kind. It is well known that under such a limiting process two situations may occur:

i) The limiting position of the system exists, thus one can talk about the continuous or discontinuous dependence of the solutions on the parameter.

Consider, for instance, the following one-parameter scalar autonomous Caushy problem

$$x'' + f(x,p) = 0, \;\; x(0) = \alpha, \;\; x'(0) = \beta,$$

when the parameter p takes large values (and tends to $+\infty$). Under the assumption that f satisfies some monotonicy conditions and it approaches a certain function g as the parameter p tends to $+\infty$, a geometric argument is used in the literature (see, e.g., Elias and Gingold [7]) to show, among others, that if the initial values lie in a suitable domain on the plane, then the solution approximates (in the C^1-sense) the corresponding solution of the limiting equation. The same behavior have the periods (in case of periodic solutions) and the escape times (in case of non-periodic solutions). Donal O' Regan in his informative book [15], p. 14, presents a problem involving a second order differential equation, when the boundary conditions are of the form $y(0) = a$ (fixed) and $y(1) = \frac{a}{n}$, when n is large enough. It is shown that for a delay equation of the form

$$\varepsilon \dot{x}(t) + x(t) = f(x(t-1)),$$

when f satisfies some rather mild conditions, there exists a periodic solution which is close to the square wave corresponding to the limiting (as $\varepsilon \to 0^+$) difference equation:

$$x(t) = f(x(t-1)).$$

Similarly, as it is shown in Ch. 10 of the book of Ferdinand Verhulst [22], the equation

$$x'' + x = \varepsilon f(x, x', \varepsilon), \;\; (x, x') \in D \subseteq \mathbb{R}^2 \tag{1.1}$$

($\varepsilon > 0$ and small) associated with the initial conditions

$$x(0) = a(\varepsilon), \;\; x'(0) = 0,$$

under some conditions on f, has a periodic solution $x(t;\varepsilon)$ satisfying

$$\lim_{\varepsilon\to 0+} x(t;\varepsilon) = a(0)\cos t.$$

Notice that the limiting value $a(0)\cos t$ is the solution of (1.1) when $\varepsilon = 0$.

ii) There exist some coefficients of the system which vanish, or tend to infinity, as the parameter approaches a liming value. In this case we can not formulate a limiting equation; however we have an asymptotic approximate system for values of the parameter which are close to the limiting value. The advantage of this situation is that in many circumstances it is possible to have information on the solutions of the limiting systems and, moreover, to compute (in closed form) the so-called approximate solutions.

A simple prototype of this situation is, for instance, the differential equation

$$\varepsilon\frac{d^2u}{dt^2} + 2\frac{du}{dt} + u = 0, \quad t > 0,$$

subject to the initial values

$$u(0) = a, \quad \frac{du}{dt} = b + \frac{\gamma}{\varepsilon}, \tag{1.2}$$

discussed in the literature and especially in the classic detailed book due to Donald R. Smith [19], p. 134. Here the parameter ε is small enough and it approaches zero.

A more general situation, which we will discuss later in Section 5, is an equation of the form

$$x'' + [a_1(t) + a_2(t)p^\nu]x' + [b_1(t) + b_2(t)p^\mu]x + a_0p^m x\sin(x) = 0, \quad t > 0 \tag{1.3}$$

associated with the initial values

$$x(0;p) = \delta_1 + \delta_2 p^\sigma, \quad x'(0;p) = \eta_1 + \eta_2 p^\tau. \tag{1.4}$$

The entities μ, ν, m, σ and τ are real numbers and p is a large parameter.

The previous two problems have the general form

$$x''(t) + a(t;p)x'(t) + b(t;p)x(t) + f(t, x(t);p) = 0, \quad t > 0, \tag{1.5}$$

where the parameter p is large enough, while the initial values are of the form

$$x(0;p) = x_0(p), \quad x'(0;p) = \bar{x}_0(p). \tag{1.6}$$

It is well known that the Krylov-Bogoliubov method was developed in the 1930's to handle situations described by second order ordinary differential equations of the form (1.5) motivated by problems in mechanics of the type generated by the Einstein equation for Mercury.

This approach, which was applied to various problems presented in [19], is based on the so called O'Malley [12], [13] and Hoppensteadt [8] method. According to this method (in case f does not depend on x) we seek an additive decomposition of the solution x of (1.5) in the form

$$x(t;p) \sim U(t;p) + U^*(\tau;p),$$

where $\tau := tp$ is the large variable and U, U^* are suitable functions, which are to be obtained in the form of asymptotic expansions, as

$$U(t;p) = \sum_{k=0}^{\infty} U_k(t)p^{-k}$$

and

$$U^*(t;p) = \sum_{k=0}^{\infty} U_k^*(t)p^{-k}.$$

After the coefficients U_k and U_k^* are determined we define the remainder

$$R_N := R_N(t;p)$$

by the relation

$$x(t;p) = \sum_{k=0}^{\infty}[U_k(t) + U_k^*(t)]p^{-k} + R_N(t;p)$$

and then obtain suitable C^1 estimates of R_N (see, [19], p. 146). This method is applied when the solutions admit initial values as in (1.2). For the general O'Malley-Hoppensteadt construction an analogous approach is followed elsewhere, see [19], p. 117. In the book due to R.E. O' Malley [14] an extended exhibition of the subject is given. The central point of the method is to obtain approximation of the solution, when the system depends on a small parameter tending to zero, (or equivalently, on a large parameter tending to $+\infty$). The small parameter ϵ is used in some of these cases and the functions involved are smooth enough to guarantee the existence and uniqueness of solutions.

In the literature one can find a great number of works dealing with singular boundary value problems, performing a set of different methods. For instance, the work due to Kadalbajoo and Patidar [10] presents a (good background and a very rich list of references on the subject, as well as a) deep survey of numerical techniques used in many circumstances to solve singularly perturbed ordinary differential equations. Also, in [21] a problem of the form

$$-\varepsilon u''(t) + p(t)u'(t) + q(t)u(t) = f(x), \;\; u(a) = \alpha_0, \;\; u(b) = \alpha_1,$$

is discussed, by using splines fitted with delta sequences as numerical strategies for the solution. See, also, [20]. A similar problem is discussed in [5], where the authors use a fourth-order finite-difference method. In [11] a problem of the form

$$\varepsilon y''(t) + [p(y(x))]' + q(x, y(x)) = r(x), \;\; y(a) = \alpha, \;\; y(b) = \beta,$$

is investigated by reducing it into an equivalent first order initial value problem and then by applying an appropriate non-linear one-step explicit scheme. In [17], where a problem of the form

$$\varepsilon y''(t) = f(x, y, x'), \;\; y(a) = y_a, \;\; y(b) = y_b,$$

is discussed, a smooth locally-analytical method is suggested. According to this method first the author considers nonoverlapping intervals and then linearize the ordinary differential equation around a fixed point of each interval. The method applies by imposing some continuity conditions of the solution at the two end points of each interval and of its first-order derivative at the common end point of two adjacent intervals.

A similar problem as above, but with boundary conditions of the form

$$y'(0) - ay(0) = A, \;\; y'(1) + by(1) = B,$$

is presented in [1], where a constructive iteration procedure is provided yielding an alternating sequence which gives pointwise upper and lower bounds on the solution.

The so called method of *small intervals* is used in [23], where the same problem as above is discussed but with impulses. In some other works, as e.g. [4], [2] (see also the references therein) two-point boundary value problems concerning third order differential equations are investigated, when the conditions depend on the (small) parameter ε. The methods used in these problems are mainly computational.

In this work our contribution to the subject is to give (assumptions and) information on the existence and the form of a C^1-approximate solution $\tilde{x}(t;p)$ of the ordinary differential equation (1.5), when the parameter p tends to $+\infty$, but by following a different approach: We suggest a smooth transformation of the time through which the equation (1.5) looks like a perturbation of an equation of the same order and with constant coefficients. The latter is used to get the approximate solution of the original equation without using the Sturm transformation. Furthermore, these arguments permit us to provide information on the estimates

$$x(t;p) - \tilde{x}(t;p)$$

and

$$\frac{d}{dt}\Big(x(t;p) - \tilde{x}(t;p)\Big),$$

as p tends to $+\infty$, uniformly for t in compact intervals. To handle the "size" of the approximation we introduce and use a kind of measure of boundedness of a function, which we term *the growth index.*

Our approach differs from that one used (recently) in [3] for the equation of the form

$$x'' + (p^2 q_1(t) + q_2(t))x = 0, \tag{1.7}$$

when p approaches $+\infty$. In [3] the authors suggest a method to approximate the solutions of (1.7) satisfying the boundary conditions of the form

$$x(0) = x_0, \ \ x(1) = mx(\xi). \tag{1.8}$$

To do that they provide an approximation of the equation, and then (they claim that) as the parameter p tends to $+\infty$, the solution of the old equation approaches the solution of the new one. And this fact is an implication of the following claim:

If a function $\delta(p)$, $p \geq 0$ satisfies $\delta(p) = o(p^{-2})$, as $p \to +\infty$, then the solution of the equation

$$v''(z;p) + v(z;p) = \delta(p)v(z;p),$$

approaches the solution of the equation

$$v''(z;p) + v(z;p) = 0.$$

However, as one can easily see, this is true only when $v(z;p) = O(p^r)$, as $p \to +\infty$, uniformly for all z, for some $r \in (0,2)$. Therefore in order to handle such cases more information on the solutions are needed.

This work is organized as follows:

In Section 2 we introduce the meaning of the growth index of a function and some useful characteristic properties of it. The basic assumptions of our problem and the auxiliary transformation of the original equation (1.5) is presented in Section 3, while in Sections 4 and 6 we give results on the existence of C^1-approximate solutions of the initial value problem (1.3)-(1.6). In Section 4 we consider equation (1.5) when the coefficient $b(t;p)$ takes (only) positive values and in Section 6 we discuss the case when $b(t;p)$ takes (only) negative values. Illustrative examples are given in Sections 5 and 7. Section 8 of the work is devoted to the approximate solutions of the boundary value problem

$$x''(t) + a(t;p)x'(t) + b(t;p)x(t) + f(t, x(t);p) = 0, \ \ t \in (0,1), \tag{1.9}$$

associated with the boundary conditions of Dirichlet type

$$x(0;p) = x_0(p), \ \ x(1;p) = x_1(p), \tag{1.10}$$

where the boundary values depend on the parameter p, as well. Here we use the (fixed point theorem of) Nonlinear Alternative to show the existence of solutions and then we present the approximate solutions. Some applications of these results are given in Section 9. In Section 10 we investigate the existence of C^1-approximate solutions of equation (1.9) associated with the boundary conditions (1.8). Again, the Nonlinear Alternative is used for the existence of solutions and then C^1-approximate solutions are given. An application of this result is given in the last section 11.

2. The growth index of a function

Before proceeding to the discussion of the main problem it is convenient to present some auxiliary facts about the growth of a real valued function f defined in a neighborhood of $+\infty$. For such a function we introduce an index, which, in a certain sense denotes the critical point at which the function stays in a real estate as the parameter tends to $+\infty$, relatively to a positive and unbounded function $E(\cdot)$. This meaning, which we term *the growth index* of f, will help us to calculate and better understand the approximation results. More facts about the growth index of functions will be published in a subsequent work.

All the (approximation) results of this work are considered with respect to a basic positive function $E(p)$, $p \geq 0$, as, e.g., $E(p) := \exp(p)$, or in general $E(p) := \exp^{(n)}(p)$, for all integers n. Here $\exp^{(0)}(p) := p$, and $\exp^{(-k)}(p) := \log^{(k)}(p)$, for all positive integers k. Actually, the function $E(p)$ denotes the level of convergence to $+\infty$ of a function h satisfying $h(p) = O((E(p))^{\mu})$, as $p \to +\infty$. The latter stands for the well known big-O symbol.

From now on we shall keep fixed such a function $E(p)$. To this function corresponds the set

$$\mathcal{A}_E := \{h : [0, +\infty) : \exists b \in \mathbb{R} : \limsup_{p \to +\infty} (E(p))^b |h(p)| < +\infty\}.$$

Then, for any $h \in \mathcal{A}_E$ we define the set

$$\mathcal{N}_E(h) := \{b \in \mathbb{R} : \limsup_{p \to +\infty} (E(p))^b |h(p)| < +\infty\}.$$

It is obvious that the set $\mathcal{N}_E(h)$ is a connected interval of the real line, whenever it is nonvoint[1]. In this case a very characteristic property of

[1]For instance, for the function $E(p) := p^2$ and a function like $h(p) := e^{\rho p}$, $\lambda > 0$ the set $\mathcal{N}_E(h)$ is empty.

the function $h \in \mathcal{A}_E$ is the quantity

$$\mathcal{G}_E(h) := \sup \mathcal{N}_E(h),$$

which we call *the growth index of h with respect to E.* To save space in the sequel the expression *with respect to E* will not be used.

The simplest case for the growth index can be met in case of the logarithm of the absolute value of an (entire complex valued function) of finite order. Indeed, if F is such a function, its order is defined as the least of all reals α such that

$$|F(z)| \leq \exp(|z|^{\alpha}),$$

for all complex numbers z. Now, the function $f(p) := \log |F(p+i0)|$ satisfies

$$\limsup_{p \to +\infty} (E(p))^b |f(p)| < +\infty$$

for all $b \leq -\alpha$, with respect to the level $E(p) := p$. Thus we have $\mathcal{G}_E(f) \geq -\alpha$.

More generally, the growth index of a function h such that $h(p) = O(p^k)$, as $p \to +\infty$, for some $k \in \mathbb{R}$, satisfies $\mathcal{G}_E(h) \geq -k$. Also, we observe that, if it holds

$$\mathcal{G}_E(h) > b,$$

then the function h satisfies

$$h(p) = O\Big([E(p)]^{-b}\Big), \text{ as } p \to +\infty,$$

or equivalently,

$$|h(p)| \leq K(E(p))^{-b},$$

for all p large enough and for some $K > 0$, not depending on p.

We present a list of characteristic properties of the growth index; some of them will be useful in the sequel.

Proposition 2.1. *If h_1, h_2 are elements of the class $\mathcal{A}_E$, then their product $h_1 h_2$ also is an element of the same space $\mathcal{A}_E$ and moreover it holds*

$$\mathcal{G}_E(h_1 h_2) \geq \mathcal{G}_E(h_1) + \mathcal{G}_E(h_2).$$

Proof. Given $h_1, h_2 \in \mathcal{A}_E$, take any b_1, b_2 such that $b_j < \mathcal{G}_E(h_j)$, $j = 1, 2$. Thus we have

$$\limsup_{p \to +\infty} (E(p))^{b_1} |h_1(p)| < +\infty \text{ and } \limsup_{p \to +\infty} (E(p))^{b_2} |h_2(p)| < +\infty$$

and therefore

$$\limsup_{p\to+\infty}(E(p))^{b_1+b_2}|h_1(p)h_2(p)| \le \limsup_{p\to+\infty}(E(p))^{b_1}|h_1(p)| \times \limsup_{p\to+\infty}(E(p))^{b_2}|h_2(p)| < +\infty.$$

This shows, first, that $h_1h_2 \in \mathcal{A}_E$ and, second, that $\mathcal{G}_E(h_1h_2) \ge b_1+b_2$. The latter implies that

$$\mathcal{G}_E(h_1h_2) \ge \mathcal{G}_E(h_1) + \mathcal{G}_E(h_2).$$

□

Lemma 2.2. *Consider the functions $h_1, h_2, \cdots, h_n$ in $\mathcal{A}_E$. Then, for all real numbers $a_j > 0$, the function $\sum_{j=1}^n a_jh_j$ belongs to $\mathcal{A}_E$ and moreover it satisfies*

$$\mathcal{G}_E\Big(\sum_{j=1}^n a_jh_j\Big) = \min\{\mathcal{G}_E(h_j) : j = 1, 2, \cdots, n\}. \tag{2.1}$$

Proof. The fact that $\sum_{j=1}^n a_jh_j$ is an element of $\mathcal{A}_E$ is obvious. To show the equality in (2.1), we assume that the left side of (2.1) is smaller than the right side. Then there is a real number N such that

$$\mathcal{G}_E\Big(\sum_{j=1}^n \alpha_jh_j\Big) < N < \min\{\mathcal{G}_E(h_j) : j = 1, 2, \cdots, n\}.$$

Thus, on one hand we have

$$\limsup_{p\to+\infty}\sum_{j=1}^n a_j\big(E(p)\big)^N|h_j(p)| = \limsup_{p\to+\infty}(E(p))^N\Big(\sum_{j=1}^n a_j|h_j(p)|\Big) = +\infty \tag{2.2}$$

and on the other hand it holds

$$\limsup_{p\to+\infty}(E(p))^N|h_j(p)| < +\infty, \quad j = 1, 2, ..., n.$$

The latter implies that

$$\limsup_{p\to+\infty}\sum_{j=1}^n a_j\big(E(p)\big)^N|h_j(p)| \le \sum_{j=1}^n a_j \limsup_{p\to+\infty}\big(E(p)\big)^N|h_j(p)| < +\infty,$$

contrary to (2.2).

If the right side of (2.1) is smaller than the left one, there is a real number N such that

$$\mathcal{G}_E\Big(\sum_{j=1}^{n} a_j h_j\Big) > N > \min\{\mathcal{G}_E(h_j) : j = 1, 2, \cdots, n\}.$$

Thus, on one hand we have

$$\limsup_{p\to+\infty}(E(p))^N \sum_{j=1}^{n} a_j |h_j(p)| < +\infty \tag{2.3}$$

and on the other hand it holds

$$\limsup_{p\to+\infty}(E(p))^N |h_{j_0}(p)| = +\infty,$$

for some $j_0 \in \{1, 2, ..., n\}$. The latter implies that

$$\limsup_{p\to+\infty}(E(p))^N \sum_{j=1}^{n} a_j |h_j(p)| \geq \limsup_{p\to+\infty} a_{j_0}(E(p))^N |h_{j_0}(p)| = +\infty,$$

contrary to (2.3). □

The growth index of a function denotes the way of convergence to zero at infinity of the function. Indeed, we have the following:

Proposition 2.3. *For a given function $h : [r_0, +\infty) \to \mathbb{R}$ it holds*

$$\mathcal{G}_E(h) = \sup\{r \in \mathbb{R} : \limsup_{p\to+\infty}(E(p))^r |h(p)| = 0\}.$$

Proof. If $b > \mathcal{G}_E(h)$, then

$$\limsup_{p\to+\infty}(E(p))^b |h(p)| = +\infty.$$

Thus, it is clearly enough to show that for any real b with $b < \mathcal{G}_E(h)$ it holds

$$\limsup_{p\to+\infty}(E(p))^b |h(p)| = 0.$$

To this end consider real numbers $b < b_1 < \mathcal{G}_E(h)$. Then we have

$$\limsup_{p\to+\infty}(E(p))^{b_1} |h(p)| =: K < +\infty$$

and therefore

$$\begin{aligned}\limsup_{p\to+\infty}(E(p))^b |h(p)| &= \limsup_{p\to+\infty}(E(p))^{(b-b_1)} \limsup_{p\to+\infty}(E(p))^{b_1} |h(p)| \\ &= \limsup_{p\to+\infty}(E(p))^{(b-b_1)} K = 0.\end{aligned}$$

□

In the sequel the choice of a variable t uniformly in compact subsets of a set U will be denoted by

$$t \in Co(U).$$

Especially we make the following:

Notation 2.4. *Let $H(t;p)$ be a function defined for $t \in S \subseteq \mathbb{R}$ and p large enough. In the sequel in case we write*

$$H(t;p) \simeq 0, \quad as \ p \to +\infty, \quad t \in Co(S),$$

we shall mean that given any compact set $I \subseteq S$ and any $\varepsilon > 0$ there is some $p_0 > 0$ such that

$$|H(t;p)| \leq \varepsilon,$$

for all $t \in I$ and $p \geq p_0$.

Also, keeping in mind Proposition 2.3 we make the following:

Notation 2.5. *Again, let $h(t;p)$ be a function defined for $t \in S \subseteq \mathbb{R}$ and p large enough. Writing*

$$\mathcal{G}_E(h(t;p)) \geq b, \quad t \in Co(S),$$

we shall mean that, for any $m < b$, it holds

$$(E(p))^m h(t;p) \simeq 0, \quad as \ p \to +\infty, \quad t \in Co(S).$$

3. Transforming equation (1.5)

In this section our purpose is to present a transformation of the one-parameter family of differential equations of the form (1.5), to a second order ordinary differential equation having constant coefficients.

Let $T_0 > 0$ be fixed and define $I := [0, T_0)$. Assume that the functions a, b, f are satisfying the following:

Condition 3.1. *For all large p the following statements are true:*

(1) The function $f(\cdot, \cdot; p)$ is continuous,

(2) $a(\cdot; p) \in C^1(I)$,

(3) There exists some $\theta > 0$ such that $|b(t;p)| \geq \theta$, for all t and all p large. Also assume that $b(\cdot; p) \in C^2(I)$ and $sign[b(t;p)] =: c =$ constant, for all $t \in I$.

The standard existence theory ensures that if Condition 3.1 holds, then equation (1.5) admits at least one solution defined on a (nontrivial) maximal interval of the form $[0, T) \subseteq [0, T_0)$.

To proceed, fix any $\hat{t} \in (0, T)$ and, for a moment, consider a strictly increasing one parameter C^2- mapping

$$v = v(t;p): \ [0, \hat{t}] \longrightarrow [0, v(\hat{t}, p)] =: J$$

with $v(0;p) = 0$. Let $\phi(\cdot;p)$ be the inverse of $v(\cdot;p)$. These functions will be defined later. If $x(t;p)$, $t \in [0, \hat{t}]$ is a solution of (1.5), define the transformation

$$S_p: \ x(\cdot;p) \to S_p x(\cdot;p): \ Graph(x(\cdot;p))\Big(\subseteq C([0,\hat{t}], \mathbb{R})\Big) \to C(J, \mathbb{R}),$$

where

$$(S_p x(\cdot;p))(v) =: y(v;p) := \frac{x(t;p)}{Y(t;p)} = \frac{x(\phi(v;p);p)}{Y(\phi(v;p);p)}, \ v \in J. \qquad (3.1)$$

Here $Y(\cdot;p)$, which will be specified later, is a certain C^2-function, depending on the parameter p. We observe that

$$x'(t;p) = Y'(t;p)y(v;p) + Y(t;p)v'(t;p)y'(v;p), \ t \in [0, \hat{t}]$$

and

$$\begin{aligned} x''(t;p = {} & Y''(t;p)y(v;p) + 2Y'(t;p)v'(t;p)y'(v;p) \\ & + Y(t;p)v''(t;p)y'(v;p) \\ & + Y(t;p)(v'(t;p))^2 y''(v;p), \ t \in [0, \hat{t}]. \end{aligned}$$

Then, equation (1.5) is transformed into the equation

$$y''(v;p) + A(t;p)y'(v;p) + B(t;p)y(v;p) + g(t;p) = 0, \ v \in J, \qquad (3.2)$$

where the one-parameter functions A, B and g are defined as follows:

$$A(t;p) := \frac{2Y'(t;p)v'(t;p) + Y(t;p)v''(t;p) + a(t;p)Y(t;p)v'(t;p)}{Y(t;p)(v'(t;p))^2},$$

$$B(t;p) := \frac{Y''(t;p) + a(t;p)Y'(t;p) + b(t;p)Y(t;p)}{Y(t;p)(v'(t;p))^2},$$

$$g(t;p) := \frac{f(t, Y(t;p)y(v;p);p)}{Y(t;p)(v'(t;p))^2}.$$

We will specify the new functions v and Y. To get the specific form of the function $v(\cdot;p)$ we set

$$v'(t;p) = \sqrt{cb(t;p)}, \ t \in I, \qquad (3.3)$$

where, recall that,

$$c = sign[b(t;p)], \ t \in I.$$

In order to have $v(t;p) \geq v(0;p) = 0$, it is enough to get

$$v(t;p) = \int_0^t \sqrt{cb(s;p)}ds, \ t \in [0, \hat{t}]. \qquad (3.4)$$

Setting the coefficient $A(t;p)$ in (3.2) equal to zero, we obtain

$$2Y'(t;p)v'(t;p)+Y(t;p)v''(t;p)+a(t;p)Y(t;p)v'(t;p)=0,\quad t\in[0,\hat{t}], \tag{3.5}$$

which, due to (3.3), implies that

$$Y'(t;p)+\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)Y(t;p)=0,\quad t\in[0,\hat{t}]. \tag{3.6}$$

We solve this equation, by integration and obtain

$$Y(t;p)=Y(0;p)\exp\Big(\int_0^t[-\frac{b'(s;p)}{4b(s;p)}-\frac{a(s;p)}{2}]ds\Big),$$

namely,

$$Y(t;p)=\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}\exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big),\quad t\in[0,\hat{t}], \tag{3.7}$$

where, without lost of generality, we have set $Y(0;p)=1$.

From (3.6) it follows that

$$\frac{Y'(t;p)}{Y(t;p)}=-\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}, \tag{3.8}$$

from which we get

$$\begin{aligned}Y''(t;p)=&-Y'(t;p)\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)\\&-Y(t;p)\Big(\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}+\frac{a'(t;p)}{2}\Big).\end{aligned} \tag{3.9}$$

Then, from relations (3.6), (3.8) and (3.9) we obtain

$$\begin{aligned}&Y''(t;p)+a(t;p)Y'(t;p)+b(t;p)Y(t;p)\\&\quad=-Y'(t;p)\Big(\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}\Big)\\&\quad-Y(t;p)\Big(\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}+\frac{a'(t;p)}{2}-b(t;p)\Big)\\&\quad=Y(t;p)\Big[\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)\Big(\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}\Big)\\&\quad-\frac{b(t;p)b''(t;p)-(b'(t;p))^2}{4(b(t;p))^2}-\frac{a'(t;p)}{2}+b(t;p)\Big].\end{aligned}$$

Hence, the expression of the function B appeared in (3.2) takes the form

$$\begin{aligned}
B(t;p) = & -\frac{1}{(v'(t;p))^2}\Big[\Big(\frac{b'(t;p)}{4b(t;p)}+\frac{a(t;p)}{2}\Big)\Big(\frac{b'(t;p)}{4b(t;p)}-\frac{a(t;p)}{2}\Big) \\
& -\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}-\frac{a'(t;p)}{2}+b(t;p)\Big] \\
= & \frac{1}{cb(t;p)}\Big[\Big(\frac{[b'(t;p)]^2}{16[b(t;p)]^2}-\frac{[a(t;p)]^2}{4}\Big) \\
& -\frac{b(t;p)b''(t;p)-[b'(t;p)]^2}{4[b(t;p)]^2}-\frac{a'(t;p)}{2}+b(t;p)\Big] \\
= & \frac{5}{16c}\frac{[b'(t;p)]^2}{(b(t;p))^3}-\frac{1}{4c}\frac{[a(t;p)]^2}{b(t;p)} \\
& -\frac{1}{4c}\frac{b''(t;p)}{[b(t;p)]^2}-\frac{a'(t;p)}{2cb(t;p)}+\frac{1}{c}.
\end{aligned}$$

Therefore equation (3.2) becomes

$$y''(v;p)+cy(v;p)=C(t,y(v;p);p)y(v;p), \quad v\in J, \tag{3.10}$$

where

$$\begin{aligned}
C(t,u;p) := & -\frac{5c}{16}\frac{[b'(t;p)]^2}{[b(t;p)]^3}+c\frac{[a(t;p)]^2}{4b(t;p)} \\
& +\frac{c}{4}\frac{b''(t;p)}{[b(t;p)]^2}+c\frac{a'(t;p)}{2b(t;p)}-c\frac{f(t,Y(t;p)u)}{b(t;p)Y(t;p)u}.
\end{aligned}$$

(Recall that $c=\pm1$, thus $c^2=1$.) The expression of the function $C(t,u;p)$ might assume a certain kind of singularity for $u=0$, but, as we shall see later, due to condition (3.13), such a case is impossible.

Therefore we have proved the *if* part of the following theorem:

Theorem 3.2. *Consider the differential equation* (1.5) *and assume that Condition 3.1 keeps in force. Then, a function* $y(v;p)$, $v\in J$ *is a solution of the differential equation* (3.10), *if and only if, the function*

$$x(t;p)=(S_p^{-1}y(\cdot;p))(t)=Y(t;p)y(v(t;p);p), \quad t\in[0,\hat{t}]$$

is a solution of (1.5). *The quantities* Y *and* v *are functions defined in* (3.7) *and* (3.4) *respectively.*

Proof. It is enough to prove the *only if* part. From the expression of $x(t;p)$ we get

$$x'(t;p)=Y'(t;p)y(v(t;p);p)+Y(t;p)v'(t;p)y'(v(t;p);p)$$

and

$$x''(t;p) = Y''(t;p)y(v(t;p);p) + 2Y'(t;p)v'(t;p)y'(v(t;p);p) \\ + Y(t;p)v''(t;p)y'(v(t;p);p) + Y(t;p)(v'(t;p))^2y''(v(t;p);p).$$

Then, by using (3.5), (3.2) and the expression of the quantity $B(t;p)$, we obtain

$$x''(t) + a(t;p)x'(t) + b(t;p)x(t) + f(t,x(t);p) \\ = Y(t;p)(v'(t;p))^2\Big[y''(v(t;p);p) + B(t;p)y(v(t;p);p) + g(t;p)\Big] = 0.$$

□

To proceed we make the following condition:

Condition 3.3. *For each* $j = 1, 2, ..., 5$, *there is a nonnegative function* $\Phi_j \in \mathcal{A}_E$, *such that, for all* $t \in [0, T)$, $z \in \mathbb{R}$ *and large* p, *the inequalities*

$$\begin{aligned} |b'(t;p)|^2 &\leq \Phi_1(p)|b(t;p)|^3, \\ |b''(t;p)| &\leq \Phi_2(p)|b(t;p)|^2, \end{aligned} \tag{3.11}$$

$$\begin{aligned} |a(t;p)|^2 &\leq \Phi_3(p)|b(t;p)|, \\ |a'(t;p)| &\leq \Phi_4(p)|b(t;p)|, \end{aligned} \tag{3.12}$$

$$|f(t,z;p)| \leq \Phi_5(p)|zb(t;p)| \tag{3.13}$$

hold.

If Condition *3.3* is true, then we have the relation

$$\left|\frac{b'(0;p)}{b(0;p)}\right| \leq \sqrt{\Phi_1(p)b(0;p)}, \tag{3.14}$$

as well as the estimate

$$\begin{aligned} |C(t,u;p)| &\leq \frac{5}{16}\Phi_1(p) + \frac{1}{4}\big(\Phi_2(p) + \Phi_3(p)\big) + \frac{1}{2}\Phi_4(p) + \Phi_5(p) \\ &=: P(p), \end{aligned} \tag{3.15}$$

for all $t \in [0, T)$ and p large enough.

4. Asymptotic approximation of the Initial Value Problem (1.5)-(1.6) in case $c = +1$

The previous facts will now help us to provide useful information on the asymptotic properties of the solutions of equation (1.5) having initial values which depend on the large parameter p, and are of the form (1.6).

In this subsection we assume that $c = +1$, thus the last requirement in Condition *3.1* keeps in force with $b(t;p) > 0$, for all $t \geq 0$ and p large enough.

As we have shown above, given a solution $x(t;p)$, $t \in [0,\hat{t}]$ of (1.5) the function $y(v;p)$, $v \in J$ defined in (3.1) solves equation (3.10) on the interval J. (Recall that J is the interval $[0, v(\hat{t};p)]$.) We shall find the images of the initial values (1.6) under this transformation.

First we note that

$$y(0;p) =: y_0(p) = \frac{x(0;p)}{Y(0;p)} = x(0;p) = x_0(p). \tag{4.1}$$

Also, from the fact that

$$x'(0;p) = Y'(0;p)y(0;p) + Y(0;p)v'(0;p)y'(0;p)$$

and relation (3.6) we obtain

$$y'(0;p) =: \hat{y}_0(p) = \frac{1}{\sqrt{b(0;p)}}\Big[\bar{x}_0(p) + \Big(\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\Big)x_0(p)\Big]. \tag{4.2}$$

Consider the solution $w(v;p)$ of the homogeneous equation

$$w'' + w = 0 \tag{4.3}$$

having the same initial values (4.1)-(4.2) as the function $y(\cdot;p)$. This requirement implies that the function $w(v;p)$ has the form

$$w(v;p) = c_1(p)\cos v + c_2(p)\sin v, \quad v \in \mathbb{R},$$

for some real numbers $c_1(p), c_2(p)$, which are uniquely determined by the initial values of $y(\cdot;p)$, namely $c_1(p) = y_0(p)$ and $c_2(p) = \hat{y}_0(p)$. Then the difference function

$$R(v;p) := y(v;p) - w(v;p), \tag{4.4}$$

satisfies

$$R(0;p) = R'(0;p) = 0,$$

and moreover

$$\begin{aligned} R''(v;p) + R(v;p) = {} & C(t, y(v;p);p)R(v;p) \\ & + C(t, y(v;p);p)w(v;p), \quad v \in J. \end{aligned} \tag{4.5}$$

Since the general solution of (4.3) having zero initial values is the zero function, applying the variation-of-constants formula in (4.5) we obtain

$$\begin{aligned} R(v;p) = {} & \int_0^v K(v,s)C(s;p;y(s;p))w(s;p)ds + \\ & + \int_0^v K(v,s)C(s;p;y(s;p))R(s;p)ds, \end{aligned} \tag{4.6}$$

where

$$K(v,s) = \sin(v-s).$$

Observe that

$$\int_0^v |\sin(v-s)w(s;p)|ds \le (|c_1(p)|+|c_2(p)|)v =: \gamma(p)v, \;\; v\in J$$

and therefore

$$|R(v;p)| \le P(p)\gamma(p)v + P(p)\int_0^v |R(s;p)|ds.$$

Applying Gronwall's inequality we obtain

$$|R(v;p)| \le \gamma(p)(e^{P(p)v}-1). \tag{4.7}$$

Differentiating $R(v;p)$ (with respect to v) in (4.6) and using (4.7), we see that the quantity $|R'(v;p)|$ has the same upper bound as $R(v;p)$ namely, we obtain

$$\max\{|R(v;p)|, |R'(v;p)|\} \le \gamma(p)(e^{P(p)v}-1), \;\; v\in J. \tag{4.8}$$

By using the transformation S_p and relation (4.8) we get the following theorem:

Theorem 4.1. *Consider the ordinary differential equation* (1.5) *associated with the initial values* (1.6)*, where assume that* $T_0 = +\infty$ *and Condition 3.1 holds with* $c = +1$*. Assume also that there exist functions* Φ_j*,* $j = 1, 2, ..., 5$*, satisfying Condition 3.3. If* $x(t;p)$*,* $t \in [0,T)$ *is a maximally defined solution of the problem* (1.5)-(1.6)*, then it holds*

$$T = +\infty, \tag{4.9}$$

and

$$\begin{aligned}|x(t;p)-&Y(t;p)w(v(t;p);p)|\\ &\le \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac14}\exp\big(-\frac12\int_0^t a(s;p)ds\big)\\ &\times\Big(\bar{x}_0(p)+\frac{1}{\sqrt{b(0;p)}}\big[|\bar{x}_0(p)|+|x_0(p)|\big|\frac{b'(0;p)}{4b(0;p)}+\frac{a(0;p)}{2}\big|\big]\Big)\\ &\times\Big[\exp\Big(P(p)\int_0^t\sqrt{b(s;p)}ds\Big)-1\Big] =: \mathcal{M}(t;p),\end{aligned} \tag{4.10}$$

as well as

$$\Big|\frac{d}{dt}[x(t;p)-Y(t;p)w(v(t;p);p)]\Big|$$
$$\leq Y(t;p)\gamma(p)\big(e^{P(p)v(t;p)}-1\big)\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big], \tag{4.11}$$

for all $t>0$ *and* p *large enough. Here we have set*

$$w(v;p):=x_0(p)cos(v)$$
$$+\frac{1}{\sqrt{b(0;p)}}\Big(\hat{x}_0(p)+\Big(\frac{b'(0;p)}{4b(0;p)}+\frac{a(0;p)}{2}\Big)x_0(p)\Big)\sin(v),$$

and $P(p)$ *is the quantity defined in* (3.15).

Proof. Inequality (4.10) is easily implied from (4.8) and the relation

$$x(t;p)=Y(t;p)y(v(t;p);p).$$

Then property (4.9) follows from (4.10) and the fact that the solution is noncontinuable (see, e.g., [16], p. 90).

To show (4.11) observe that

$$\Big|\frac{d}{dt}[x(t;p)-Y(t;p)w(v(t;p);p)]\Big|$$
$$=\Big|\frac{d}{dt}Y(t;p)[y(v(t;p);p)-w(v(t;p);p)]\Big|$$

and therefore

$$\Big|\frac{d}{dt}[x(t;p)-Y(t;p)w(v(t;p);p)]\Big|$$
$$\leq\Big|[y(v(t;p);p)-w(v(t;p);p)]\frac{d}{dt}Y(t;p)\Big|$$
$$+\Big|Y(t;p)\frac{d}{dt}[y(v(t;p);p)-w(v(t;p);p)]\Big||$$
$$\leq\Big|R(v(t;p);p)\frac{d}{dt}Y(t;p)\Big|+\Big|Y(t;p)\frac{d}{dv}R(v(t;p);p)\frac{d}{dt}v(t;p)\Big|$$
$$\leq Y(t;p)\gamma(p)\big(e^{P(p)v(t;p)}-1\big)\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big].$$

We have used relations (3.8), (3.14) and (4.7). □

Now we present the main results concerning the existence of approximate solutions of the initial value problem (1.5) - (1.6).

The function defined by

$$\begin{aligned}\tilde{x}(t;p) : &= Y(t;p)w(v(t;p);p)\\ &= Y(t;p)[y_0(p)\cos\int_0^t\sqrt{b(s;p)}ds + \hat{y}_0(p)\sin\int_0^t\sqrt{b(s;p)}ds]\\ &= \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}\exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big)\Big\{x_0(p)\cos(v(t;p))\\ &+\frac{1}{\sqrt{b(0;p)}}\Big[\bar{x}_0(p)+\Big(\frac{b'(0;p)}{4b(0;p)}+\frac{a(0;p)}{2}\Big)x_0(p)\sin(v(t;p))\Big]\Big\}\end{aligned} \tag{4.12}$$

is the so called *approximate solution* of the problem, since, as we shall see in the sequel, this function approaches the exact solution as the parameter tends to $+\infty$. Moreover, since this function approaches the solution x in the C^1 sense, namely in a sense given in the next theorem, we shall refer to it as *a* C^1 *approximate solution.*

To make the notation short consider the *error* function

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p). \tag{4.13}$$

Then, from (4.10) and (4.11), we get

$$|\mathcal{E}(t;p)| \leq \mathcal{M}(t;p) \tag{4.14}$$

and

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &\leq Y(t;p)\gamma(p)\big(e^{P(p)v(t;p)}-1\big)\\ &\times\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big],\end{aligned} \tag{4.15}$$

respectively.

Theorem 4.2. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 4.1 keep in force and the relation*

$$\min_{j=1}^{5}\mathcal{G}_E(\Phi_j) > 0 \tag{4.16}$$

is satisfied. Moreover, we assume that

$$x_0,\ x_1 \in \mathcal{A}_E, \tag{4.17}$$

$$a(\cdot;p) \geq 0,\ \text{for all large } p, \tag{4.18}$$

as well as

$$a(t;\cdot),\ b(t;\cdot) \in \mathcal{A}_E,\ t \in C_o(\mathbb{R}^+). \tag{4.19}$$

If $\mathcal{E}(t;p)$ *is the error function defined in* (4.13) *and the relation*

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \Big[\frac{3}{4}\mathcal{G}_E(b(t;\cdot) + \min\{\mathcal{G}_E(\bar{x}_0),\\ \mathcal{G}_E(x_0) + \frac{1}{2}\mathcal{G}_E(b(t;\cdot),\\ \mathcal{G}_E(x_0) + \mathcal{G}_E(a(t;\cdot)\}\Big] =: N_0 > 0, \;\; t \in C_o(\mathbb{R}), \tag{4.20}$$

is satisfied, then we have

$$\mathcal{E}(t;p) \simeq 0, \;\; p \to +\infty, \;\; t \in C_o(\mathbb{R}^+) \tag{4.21}$$

and the growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq N_0, \;\; t \in C_o(\mathbb{R}^+). \tag{4.22}$$

In addition to the assumptions above for the functions $x_0, \bar{x}_0, a, b$ *assume the condition*

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \Big[\frac{3}{4}\mathcal{G}_E(b(t;\cdot) + \min\{\mathcal{G}_E(\bar{x}_0) + \mathcal{G}_E(a(t;\cdot),\\ \frac{1}{2}\mathcal{G}_E(b(t;\cdot) + \mathcal{G}_E(\bar{x}_0), \mathcal{G}_E(x_0) + \mathcal{G}_E(b(t;\cdot),\\ \mathcal{G}_E(x_0) + 2\mathcal{G}_E(a(t;\cdot), \frac{1}{2}\mathcal{G}_E(b(t;\cdot) + \mathcal{G}_E(x_0)\\ + \mathcal{G}_E(a(t;\cdot)\}\Big] =: N_1 > 0, \;\; t \in C_o(\mathbb{R}^+), \tag{4.23}$$

instead of (4.20)*. Then we have*

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \;\; p \to +\infty, \;\; t \in Co(\mathbb{R}^+), \tag{4.24}$$

and the growth index at infinity of the error function is such that

$$\mathcal{G}_E\big(\frac{d}{dt}\mathcal{E}(t;\cdot)\big) \geq N_1, \;\; t \in C_o(\mathbb{R}^+). \tag{4.25}$$

Proof. Due to our assumptions given $\varepsilon > 0$ small enough, we can find real numbers σ, τ, and μ, ν, close to the quantities $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(\bar{x}_0)$, $-\mathcal{G}_E(a(t;\cdot))$ and $-\mathcal{G}_E(b(t;\cdot))$ respectively, such that, as $p \to +\infty$,

$$x_0(p) = O(E(p))^{\sigma}), \;\; \bar{x}_0(p) = O((E(p))^{\tau}), \tag{4.26}$$

$$a(t;p) = O((E(p))^{\nu}), \quad \text{as } p \to +\infty, \;\; t \in C_o(\mathbb{R}^+) \tag{4.27}$$

and

$$b(t;p) = O((E(p))^{\mu}), \;\; \text{as } p \to +\infty, \;\; t \in C_o(\mathbb{R}^+), \tag{4.28}$$

as well as the relation

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) - \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \frac{\mu}{2}, \sigma + \nu\}\Big] =: N_0 - \varepsilon > 0. \tag{4.29}$$

for

Assume that (4.18) holds. We start with the proof of (4.21). Fix any $\hat{t} > 0$ and take any $N \in (0, N_0 - \varepsilon)$. Then, due to (4.16), we can let $\zeta > 0$ such that

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N + \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \frac{\mu}{2}, \sigma + \nu\}\Big].$$

Therefore we have

$$\max\{\frac{3\mu}{4} + \tau,\ \frac{5\mu}{4} + \sigma,\ \frac{3\mu}{4} + \sigma + \nu\} - \zeta < -N, \tag{4.30}$$

and, due to Lemma 2.2, it holds

$$\mathcal{G}_E(P) > \zeta,\ \mathcal{G}_E(\Phi_1) > \zeta. \tag{4.31}$$

The latter implies that there exist $K > 0$ and $p_0 > 1$ such that

$$\begin{aligned} &0 < P(p) \leq K(E(p))^{-\zeta}, \\ &0 < \Phi_1(p) \leq K(E(p))^{-\zeta}, \end{aligned} \tag{4.32}$$

for all $p \geq p_0$.

From relations (4.27), (4.28) and (4.26) it follows that there are positive real numbers $K_j, j = 1, 2, 3, 4$ such that

$$|b(t;p) \leq K_1(E(p))^{\mu}, \tag{4.33}$$

$$\begin{aligned} &|\bar{x}_0(p)| \leq K_2(E(p))^{\tau}, \\ &|x_0(p)| \leq K_3(E(p))^{\sigma}, \end{aligned} \tag{4.34}$$

$$0 \leq a(t;p) \leq K_4(E(p))^{\nu}, \tag{4.35}$$

for all $t \geq 0$ and $p \geq p_1$, where $p_1 \geq p_0$.

Also keep in mind that from Condition 3.1 we have

$$b(t;p) \geq \theta, \tag{4.36}$$

In the sequel, for simplicity, we shall denote by q the quantity $E(p)$.

Consider the function $\mathcal{M}(t;p)$ defined in (4.10). Then, due to (4.32), (3.14) and (4.33)-(4.36), for all $t \in [0, \hat{t}]$ and p with $q = E(p) \geq p_1$, we have

$$\begin{aligned} \mathcal{M}(t;p) \leq K_1^{\frac{1}{4}}\theta^{-\frac{1}{4}}q^{\frac{\mu}{4}}\Big[K_2q^{\tau} + \theta^{-\frac{1}{2}}\Big(K_2q^{\tau} + K_3q^{\sigma}\big[\frac{1}{4}(K_1K)^{\frac{1}{2}}q^{\frac{-\zeta+\mu}{2}} \\ + \frac{1}{2}K_4q^{\nu}\big]\Big)\Big]\Big(\sum_{n=1}^{+\infty}\frac{1}{n!}K^nq^{-n\zeta}t^n(K_1)^{\frac{n}{2}}q^{\frac{n\mu}{2}}\Big). \end{aligned} \tag{4.37}$$

Since the series

$$1+\sum_{n=1}^{+\infty}\frac{1}{(n+1)!}(tK)^n q^{-n\zeta}(K_1)^{\frac{n}{2}}q^{\frac{n\mu}{2}}$$

converges uniformly for t in compact sets, it admits an upper bound $K_5(\hat{t})>0$, say, on $[0,\hat{t}]$. Therefore, for all $t\in[0,\hat{t}]$ and $q=E(p)\geq p_1$, it holds

$$\sum_{n=1}^{+\infty}\frac{1}{n!}K^n q^{-n\zeta}t^n(K_1)^{\frac{n}{2}}q^{\frac{n\mu}{2}}\leq K_5(\hat{t})\hat{t}Kq^{-\zeta}(K_1)^{\frac{1}{2}}q^{\frac{\mu}{2}}.$$

So, from (4.30) and (4.37) we get

$$\begin{aligned}\mathcal{M}(t;p)&\leq K_1^{\frac{1}{4}}\theta^{-\frac{1}{4}}q^{\frac{\mu}{4}}\times\Big[(1+\theta^{-\frac{1}{2}})K_2q^{\tau}\\&\quad+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{4}(K_1K)^{\frac{1}{2}}q^{\frac{-\zeta+\mu}{2}}+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{2}K_4q^{\nu}\Big]\\&\quad\times K_5(\hat{t})\hat{t}Kq^{-\zeta}(K_1)^{\frac{1}{2}}q^{\frac{\mu}{2}}\\&=K_6q^{\frac{\mu}{4}+\tau-\zeta+\frac{\mu}{2}}+K_7q^{\frac{\mu}{4}+\sigma+\frac{-\zeta+\mu}{2}-\zeta+\frac{\mu}{2}}+K_8q^{\frac{\mu}{4}+\sigma+\nu-\zeta+\frac{\mu}{2}}\\&\leq K_6q^{-N}+K_7q^{-N-\frac{\zeta}{2}}+K_8q^{-N}<K_9q^{-N},\end{aligned}\tag{4.38}$$

for some positive constants $K_j, j=6,7,8,9$. Recall that

$$q=E(p)\geq p_1\geq p_0>1.$$

This and (4.14) complete the proof of (4.21).

Now, from the previous arguments it follows that given any $\Lambda\in(0,N)$ it holds

$$\mathcal{M}(t;p)q^{\Lambda}\leq K_9q^{-N+\Lambda}\to 0,\ \text{ as } p\to+\infty,$$

where the constant K_9 is uniformly chosen for t in the compact interval $[0,\hat{t}]$. Then from (4.14) we get

$$\mathcal{E}(t;p)q^{\Lambda}\to 0,\ \text{ as } p\to+\infty,$$

which implies that the growth index at infinity of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;p))\geq\Lambda.$$

From here we get

$$\mathcal{G}_E(\mathcal{E}(t;p))\geq N.$$

Since N is arbitrary in the interval $(0,N_0-\varepsilon)$ and ε is any small positive number, we obtain (4.22).

We proceed to the proof of (4.24).

Again, from our assumptions and (4.23), for any small enough $\varepsilon > 0$, we can choose real numbers σ, τ, and μ, ν, as above, satisfying (4.33), (4.34), (4.35), as well as

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) - \Big[\frac{3\mu}{4} + \max\{\tau + \nu,\\ \frac{\mu}{2} + \tau, \sigma + \mu, \sigma + 2\nu, \frac{\mu}{2} + \sigma + \nu\}\Big]\\ =: N_1 - \varepsilon > 0.\end{aligned} \tag{4.39}$$

Take any $N \in (0, N_1 - \varepsilon)$. Then, because of (4.39), we can choose $\zeta > 0$ such that

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N + \Big[\frac{3\mu}{4} + \max\{\tau + \nu, \frac{\mu}{2} + \tau, \sigma + \mu, \sigma + 2\nu,\\ \frac{\mu}{2} + \sigma + \nu\}\Big],\end{aligned}$$

From this relation it follows that

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) >& N + \Big[\frac{3\mu}{4} + \max\{\frac{\mu}{2} + \tau, \sigma + \mu, \frac{\mu}{2} + \sigma + \nu\}\Big]\\ =& (N + \frac{\mu}{2}) + \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \frac{\mu}{2}, \sigma + \nu\}\Big]\end{aligned}$$

and

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) >& N + \Big[\frac{3\mu}{4} + \max\{\tau + \nu, \sigma + 2\nu, \frac{\mu}{2} + \sigma + \nu\}\Big]\\ =& (N + \nu) + \Big[\frac{3\mu}{4} + \max\{\tau, \sigma + \nu, \sigma + \frac{\mu}{2}\}\Big].\end{aligned}$$

These inequalities with a double use of (4.38), with N being replaced with

$$N + \frac{\mu}{2} \text{ and } N + \nu$$

respectively imply that

$$\mathcal{M}(t;p) < K_9 q^{-N-\frac{\mu}{2}} \text{ and } \mathcal{M}(t;p) < K_9 q^{-N-\nu}.$$

Then, from (4.15), (4.32) and conditions (4.33), (4.35) it follows that there are constants K_{10}, K_{11}, K_{12} such that

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| \leq &\mathcal{M}(t;p)\Big[\frac{\sqrt{\Phi_1(p)b(t;p)}}{4}+\frac{|a(t;p)|}{2}+\sqrt{b(t;p)}\Big]\\ &\leq \mathcal{M}(t;p)[K_{10}q^{-\frac{\zeta}{2}}q^{\frac{\mu}{2}}+K_{11}q^{\nu}+K_{12}q^{\frac{\mu}{2}}]\\ &\leq K_{10}K_9q^{-N-\frac{\mu}{2}}q^{-\frac{\zeta}{2}}q^{\frac{\mu}{2}}+K_{11}K_9q^{-N-\nu}p^{\nu}\\ &\quad+K_{12}K_9q^{-N-\frac{\mu}{2}}q^{\frac{\mu}{2}}\\ &=K_{10}K_9q^{-N-\frac{\zeta}{2}}+K_{11}K_9q^{-N}+K_{12}K_9q^{-N}\\ &\leq (K_{10}+K_{11}+K_{12})q^{-N}.\end{aligned} \tag{4.40}$$

Since N is arbitrary, this relation completes the proof of (4.24).

Relation (4.25) follows from (4.40), exactly in the same way as (4.22) follows from (4.38). □

Theorem 4.3. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 4.1 and conditions* (4.17), (4.18), (4.19) *keep in force. Moreover assume that there is a measurable function* $\omega : [0,+\infty) \to [0,+\infty)$ *such that*

$$|a(t;p)| \leq \omega(t)\log(E(p)), \quad t \geq 0 \tag{4.41}$$

for p large enough. If $\mathcal{E}(t;p)$ is the error function defined in (4.13) *and the relation*

$$\begin{aligned}\min_{j=1}^{5}\lambda(\Phi_j)+\Big[\frac{3}{4}&\mathcal{G}_E(b(t;\cdot)+\min\{\mathcal{G}_E(\bar{x}_0),\mathcal{G}_E(x_0)\\ &+\frac{1}{2}\mathcal{G}_E(b(t;\cdot)),\mathcal{G}_E(x_0)\}\Big]\\ &=: M_0>0,\end{aligned} \tag{4.42}$$

holds, then we have

$$\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co([0,T(M_0))), \tag{4.43}$$

where, for any $M>0$ we have set

$$T(M) := \sup\{t>0 : \Omega(t) := \int_0^t \omega(s)ds < 2M\}. \tag{4.44}$$

In this case the growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq M_0, \quad t \in Co([0,T(M_0))). \tag{4.45}$$

Also, if (4.41) *keeps in force and the condition*

$$\begin{aligned}\min_{j=1}^{5}\lambda(\Phi_j)+\Big[\frac{3}{4}\mathcal{G}_E(b(t;\cdot))+\min\{\mathcal{G}_E(\bar{x}_0),\frac{1}{2}\mathcal{G}_E(b(t;\cdot))\\ +\mathcal{G}_E(\bar{x}_0),\mathcal{G}_E(x_0)+\mathcal{G}_E(b(t;\cdot)),\mathcal{G}_E(x_0),\\ \frac{1}{2}\mathcal{G}_E(b(t;\cdot))+\mathcal{G}_E(x_0)\}\Big]=:M_1>0\end{aligned}\tag{4.46}$$

is satisfied, then we have

$$\frac{d}{dt}\mathcal{E}(t;p)\simeq 0,\;\; p\to+\infty,\;\; t\in Co([0,T(M_1)))\tag{4.47}$$

and the growth index of the error function is such that

$$\mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;\cdot))\geq M_1,\;\; t\in Co([0,T(M_1))).\tag{4.48}$$

Proof. Let $\hat{t}\in(0,T(M_0))$ be fixed. Then from (4.42) we can choose numbers μ,σ,τ satisfying (4.33) and (4.34) and such that $-\mu,-\sigma,-\tau$ are close to $\mathcal{G}_E(b(t;\cdot))$, $\mathcal{G}_E(x_0)$ and$\mathcal{G}_E(\bar{x}_0)$, respectively and moreover

$$\Big[\frac{3\mu}{4}+\max\{\tau,\sigma+\frac{\mu}{2},\sigma\}\Big]+\frac{1}{2}\Omega(\hat{t})<\min_{j=1}^{5}\mathcal{G}_E(\Phi_j).$$

Take ζ,ν,N (strictly) positive such that

$$\begin{aligned}\Big[\frac{3\mu}{4}+\max\{\tau,\sigma+\frac{\mu}{2},\sigma+\nu\}\Big]+\frac{1}{2}\Omega(\hat{t})+N\\ <\zeta<\min_{j=1}^{5}\mathcal{G}_E(\Phi_j).\end{aligned}\tag{4.49}$$

Let $p_0>1$ be chosen so that $\log(p)\leq p^{\nu}$ and (4.41) holds, for all $p\geq p_0$. Then, due to (4.41), we have

$$|a(0;p)|\leq\omega(0)q^{\nu},\tag{4.50}$$

for all $p\geq p_0$. Recall that $q:=E(p)$.

Now we proceed as in Theorem 4.2, where, due to (4.41) and (4.50), relation (4.38) becomes

$$\begin{aligned}\mathcal{M}(t;p)&\leq K_1^{\frac{1}{4}}\theta^{-\frac{1}{4}}q^{\frac{\mu}{4}}e^{\frac{1}{2}\Omega(\hat{t})\log(q)}\\ &\times\Big[(1+\theta^{-\frac{1}{2}})K_2q^{\tau}+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{4}(K_1K)^{\frac{1}{2}}q^{\frac{-\zeta+\mu}{2}}\\ &\quad+K_3\theta^{-\frac{1}{2}}q^{\sigma}\frac{1}{2}\omega(0)\log(q)\Big]\times K_5(\hat{t})\hat{t}Kq^{-\zeta}(K_1)^{\frac{1}{2}}q^{\frac{\mu}{2}}\\ &\leq K_6q^{\frac{\mu}{4}+\tau-\zeta+\frac{\mu}{2}+\frac{1}{2}\Omega(\hat{t})}\\ &+K_7q^{\frac{\mu}{4}+\sigma+\frac{-\zeta+\mu}{2}-\zeta+\frac{\mu}{2}+\frac{1}{2}\Omega(\hat{t})}+K_8q^{\frac{\mu}{4}+\sigma+\nu-\zeta+\frac{\mu}{2}+\frac{1}{2}\Omega(\hat{t})}.\end{aligned}\tag{4.51}$$

Notice that (4.51) holds for all $q := E(p)$ with $p \geq p_0 > 1$. From this inequality and (4.49) we obtain the estimate

$$\mathcal{M}(t;p) \leq (K_6 + K_7 + K_8)q^{-N}, \tag{4.52}$$

which implies the approximation (4.43). Inequality (4.45) follows as the corresponding one in Theorem 4.2. Finally, as in Theorem 4.2, we can use the above procedure and (4.52) in order to get a relation similar to (4.40), from which (4.47) and (4.48) follow. □

5. Application to the Initial Value Problem (1.3)-(1.4)

Consider the initial value problem (1.3)-(1.4), where assume the following conditions:

(i) The function $b_1 \in C^2([0,+\infty),[0,+\infty))$ it is bounded and it has bounded derivatives.

(ii) The functions $a_1, a_2 \in C^1([0,+\infty),[0,+\infty))$ are bounded with bounded derivatives.

(iii) The function b_2 is a nonzero positive constant and, as we said previously, the exponents $\mu, \nu, m, \sigma, \tau$ of the model are real numbers.

Observe that Condition 3.3 is satisfied by choosing the following functions:

$$\Phi_1(p) = l_1 p^{-3\mu}, \;\; \Phi_2(p) = l_2 p^{-2\mu}, \;\; \Phi_3(p) = l_3 p^{2\nu-\mu},$$
$$\Phi_4(p) = l_4 p^{\nu-\mu}, \;\; \Phi_5(p) = l_5 p^{m-\mu},$$

for some positive constants $l_j,\; j = 1, 2, \cdots, 5$. It is not hard to show that the growth index of these functions with respect to the function $E(p) := p$, are

$$\mathcal{G}_E(\Phi_1) = 3\mu, \;\; \mathcal{G}_E(\Phi_2) = 2\mu, \;\; \mathcal{G}_E(\Phi_3) = -2\nu + \mu,$$
$$\mathcal{G}_E(\Phi_4) = -\nu + \mu, \;\; \mathcal{G}_E(\Phi_5) = -m + \mu.$$

In this case the results (4.21) - (4.22) and (4.24) - (4.25) keep in force with N_0 and N_1 being defined as

$$N_0 := \min\{\frac{5\mu}{4}, \frac{\mu}{4} - 2\nu, \frac{\mu}{4} - m\} - \max\{\tau, \frac{\mu}{2} + \sigma, \sigma + \nu\}$$

and

$$N_1 = \min\{\frac{5\mu}{4}, \frac{\mu}{4} - 2\nu, \frac{\mu}{4} - m\} - \max\{\tau + \nu, \mu + \sigma, \frac{\mu}{2} + \tau, \sigma + 2\nu, \frac{\mu}{2} + \sigma + \nu\},$$

respectively, provided that they are positive.

To give a specific application let us assume that the functions a_1, a_2, b_1 are constants. Then we can obtain the approximate solution of the initial value problem (1.3)-(1.4) by finding the error function.

Indeed, via (4.12), we can see that a C^1-approximate solution of problem (1.3)-(1.4) is the function defined by

$$\begin{aligned}\tilde{x}(t;p) :=& exp[-\frac{1}{2}t(a_1 + a_2p^{\nu})] \\ &\times \Big[(\delta_1 + \delta_2 p^{\sigma})\cos[t(b_1 + b_2p^{\mu})] + (b_1 + b_2p^{\mu})^{-\frac{1}{2}} \\ &\times \Big(\eta_1 + \eta_2 p^{\tau} + \frac{1}{2}(\delta_1 + \delta_2 p^{\sigma})(a_1 + a_2 p^{\nu})\Big) \\ &\times \sin[t(b_1 + b_2p^{\mu})]\Big], \quad t \geq 0.\end{aligned}$$

This approximation is uniform for t in compact intervals of the positive real axis. For instance, for the values

$$\begin{aligned} a_1 = 2,\ a_2 = \delta_2 = 0,\ \delta_1 = b_1 = b_2 = \eta_1 = \eta_2 = 1 \\ \mu = \frac{9}{10},\ \nu = \frac{1}{10},\ m < 0,\ \tau = -\frac{9}{20}, \sigma = -1, \end{aligned} \tag{5.1}$$

we can find that the growth index at infinity of the error function $\mathcal{E}(t;\cdot))$ satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq \frac{19}{40} \text{ and } \mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;\cdot)) \geq \frac{1}{40}.$$

In Figure 1 the approximate solution for the values p=50, $p = 150$ and p=250 are shown.

6. Approximate solutions of the Initial Value Problem (1.5)-(1.6) in case $c = -1$

In this section we shall discuss the IVP (1.5)-(1.6) when $c = -1$, thus we assume that $b(t;p) < 0$, for all t and large p. We shall assume throughout of this section that Condition 3.3 (given in the end of Section 3) is satisfied.

Here the function y defined in (3.1) takes initial values $y_0(p)$ and $\hat{y}_0(p)$ as in (4.1) and (4.2). We wish to proceed as in Section 4 and consider a fixed point $\hat{t} > 0$, as well as the solution

$$w(v;p) := c_1(p)e^v + c_2(p)e^{-v}, \quad v \in [0, \hat{v}]$$

of equation

$$w'' - w = 0, \tag{6.1}$$

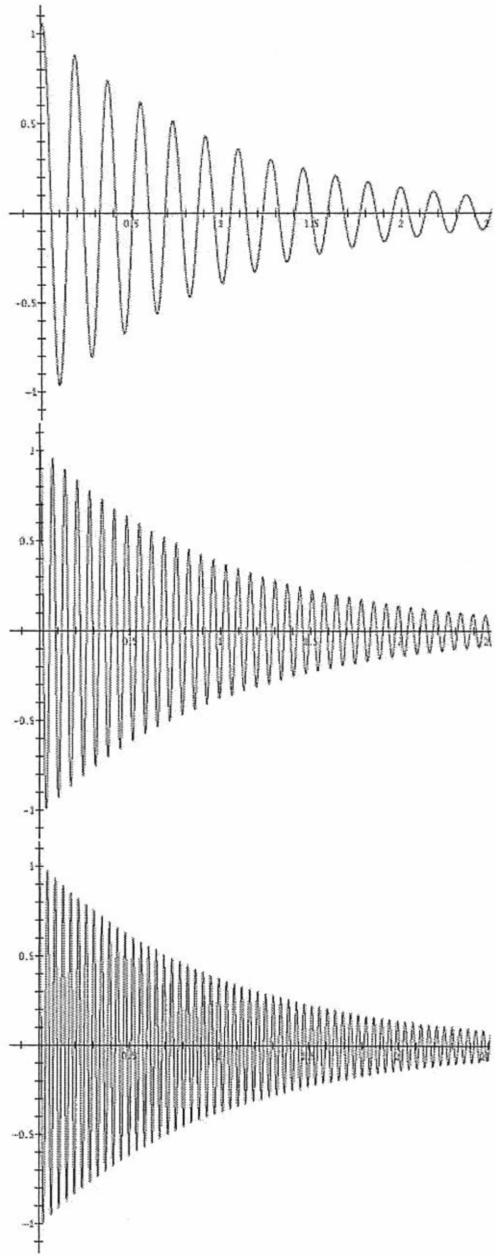

FIGURE 1. Approximate solutions of the problem (1.3)-(1.4), with the values (5.1) and when p = 50, p = 150 and p =250, respectively

associated with the same initial values as y. We have set $\hat{v} := v(\hat{t}; p)$. Thus, for $j = 1, 2$ we obtain

$$c_j(p) = \frac{1}{2}\Big[x_0(p) - \frac{(-1)^j}{\sqrt{-b(0;p)}}\Big(\bar{x}_0(p) + x_0(p)\Big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\Big]\Big)\Big]$$

and therefore it holds

$$\begin{aligned} |c_j(p)| \le \frac{1}{2}\Big[|x_0(p)| + \frac{1}{\sqrt{-b(0;p)}}\Big|\bar{x}_0(p) \\ + x_0(p)\Big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\Big]\Big|\Big] =: \kappa(p). \end{aligned} \tag{6.2}$$

Also, the difference function R defined in (4.4) satisfies (4.6) where, now, we have

$$K(v, s) = \sinh(v - s).$$

Observe that

$$\begin{aligned} \int_0^v \sinh(v-s)|w(s;p)|ds &\le \frac{|c_1(p)|}{2}\int_0^v (e^{v-s} - e^{-v+s})e^s ds \\ &+ \frac{|c_2(p)|}{2}\int_0^v (e^{v-s} - e^{-v+s})e^{-s} ds \\ &\le \frac{|c_1(p)|}{2}(ve^v - \sinh(v)) \\ &+ \frac{|c_2(p)|}{2}(\sinh(v) - ve^{-v}) \\ &\le \kappa(p)v\sinh(v) \end{aligned}$$

and therefore, for any $v \in [0, \hat{v}]$, it holds

$$\begin{aligned} |R(v;p)| &\le P(p)\kappa(p)v\sinh(v) + P(p)\int_0^v \sinh(v-s)|R(s;p)|ds \\ &\le P(p)\kappa(p)v\sinh(v) + P(p)\sinh(v)\int_0^v |R(s;p)|ds. \end{aligned}$$

Here we apply the method of proving Gronwall's inequality, but we follow a different procedure. Indeed, we set

$$F(v) := \int_0^v |R(s;p)|ds.$$

Then

$$F'(v) = |R(v;p)| \le P(p)\kappa(p)v\sinh(v) + P(p)\sinh(v)F(v)$$

and therefore

$$F'(v) - P(p)\sinh(v)F(v) \le P(p)\kappa(p)v\sinh(v).$$

Multiply both sides with the factor exp $\Big(-P(p)\cosh(v)\Big)$ and integrate from 0to v. Then we obtain

$$\begin{aligned}F(v)e^{-P(p)\cosh(v)} &\leq P(p)\kappa(p)\int_0^v s\sinh(s)e^{-P(p)\cosh(s)}ds\\ &= \kappa(p)(-ve^{-P(p)\cosh(v)} + \int_0^v e^{-P(p)\cosh(s)}ds\\ &\leq \kappa(p)v(1-e^{-P(p)\cosh(v)}).\end{aligned}$$

Therefore we have

$$\begin{aligned}|R(v;p)| &\leq P(p)\kappa(p)v\sinh(v)\\ &+ P(p)\kappa(v)v\sinh(v)(e^{P(p)\cosh(v)}-1)\\ &= P(p)\kappa(p)v\sinh(v)e^{P(p)\cosh(v)}.\end{aligned} \tag{6.3}$$

Next we observe that

$$\begin{aligned}\int_0^v \cosh(v-s)|w(s;p)|ds &\leq \frac{|c_1(p)|}{2}(ve^v+\sinh(v))\\ &+\frac{|c_2(p)|}{2}(\sinh(v)+ve^{-v})\\ &\leq \kappa(p)(v\cosh(v)+\sinh(v))\end{aligned}$$

and therefore, for any $v\in[0,\hat{v}]$, it holds

$$\begin{aligned}|R'(v;p)| &\leq P(p)\kappa(p)(v\cosh(v)+\sinh(v))\\ &+P(p)\int_0^v\cosh(v-s)|R(s;p)|ds\\ &\leq P(p)\kappa(p)(v\cosh(v)\\ &+\sinh(v))+P(p)\cosh(v)\int_0^v|R(s;p)|ds.\end{aligned}$$

Using this inequality and (6.3) we obtain

$$\begin{aligned}|R'(v;p)| &\leq P(p)\kappa(p)(v\cosh(v)+\sinh(v))\\ &+P(p)\kappa(p)e^{P(p)}v\cosh(v)(e^{P(p)(\cosh(v)-1)}-1).\end{aligned} \tag{6.4}$$

The proof of the next theorem follows as the proof of Theorem 4.1, by using (6.3), (6.4) and the expression of the functions v and Y from (3.4) and (3.7) respectively. So we omit it.

Theorem 6.1. *Consider the ordinary differential equation*(1.5) *associated with the initial values* (1.6), *where assume that the Condition 3.1 holds with c = -1. Assume also that there exist functions* Φ_j,

$j = 1, 2, ..., 5$, *satisfying* (3.11), (3.12), (3.13). *If* $x(t;p)$, $t \in [0, T)$ *is a maximally defined solution of the problem* (1.5)-(1.6), *then it holds*

$$T = +\infty,$$

and if we set

$$w(v;p) := \frac{1}{2}\sum_{j=1}^{2} e^{-(-1)^j v}\Big[x_0(p) - \frac{(-1)^j}{\sqrt{-b(0;p)}}\Big(\bar{x}_0(p) + x_0(p)\Big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\Big]\Big)\Big]$$

and

$$\mathcal{E}(t;p) := x(t;p) - Y(t;p)w(v(t;p);p),$$

then we have

$$\begin{aligned}|\mathcal{E}(t;p)| &\leq P(p)\kappa(p)\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big)\\ &\times \int_0^t \sqrt{-b(s;p)}ds \sinh\Big[\int_0^t \sqrt{-b(s;p)}ds\Big]\\ &\times \exp\big(P(p)\cosh(\int_0^t \sqrt{-b(s;p)}ds)\big) =: \mathcal{L}(t;p),\end{aligned} \tag{6.5}$$

as well as

$$\begin{aligned}\Big|\frac{d}{dt}\mathcal{E}(t;p)\Big| &\leq \mathcal{L}(t;p)\Big[\frac{\sqrt{\Phi_1(p)|b(t;p)|}}{4} + \frac{|a(t;p)|}{2}\Big]\\ &+ \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big)\sqrt{-b(t;p)}\\ &\times P(p)\kappa(p)\Big[\big(\int_0^t \sqrt{-b(s;p)}ds\big)\cosh\big(\int_0^t \sqrt{-b(s;p)}ds\big)\\ &+ \sinh\big(\int_0^t \sqrt{-b(s;p)}ds\big)\\ &+ e^{P(p)}\int_0^t \sqrt{-b(s;p)}ds \cosh\big(\int_0^t \sqrt{-b(s;p)}ds\big)\\ &\times \Big(e^{P(p)(\cosh(\int_0^t \sqrt{-b(s;p)}ds)-1} - 1\Big)\Big],\end{aligned} \tag{6.6}$$

for all $t \in I$ *and* p. *Here* P *is defined in* (3.15) *and* κ *in* (6.2).

Now we give the main results of this section.

Theorem 6.2. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 6.1 keep in force. Moreover assume that* $a(\cdot;p) \geq 0$, *for all large* p, *as well as the following properties:*

i) It holds $\sup_{t>0} b(t;p) < 0$, *uniformly for all* t *in compact sets and all large* p.

ii) It holds $\lambda(\Phi_j) > 0$, *for all* $j = 1, 2, ..., 5$.

iii) It holds $x_0, x_1 \in \mathcal{A}_E$.

Define the function

$$\tilde{x}(t;p) := \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big) \\ \times \frac{1}{2}\sum_{j=1}^{2} e^{-(-1)^j v}\Big[x_0(p) \\ - \frac{(-1)^j}{\sqrt{-b(0;p)}}\Big(\bar{x}_0(p) + x_0(p)\big[\frac{b'(0;p)}{4b(0;p)} + \frac{a(0;p)}{2}\big]\Big)\Big]. \tag{6.7}$$

Let x *be a solution of the problem and we let* $\mathcal{E}(t;p)$ *be the error function defined by*

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p).$$

a) If $a(t;\cdot) \in \mathcal{A}_E$, $t \in Co(\mathbb{R}^+)$ *holds and there is a measurable function* $z(t)$, $t \geq 0$ *such that*

$$|b(t;p)| \leq z(t)\big[\log(\log(E(p)))\big]^2, \tag{6.8}$$

for all $t \geq 0$ *and* p *large enough, then we have*

$$\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co(\mathbb{R}^+), \tag{6.9}$$

provided that the quantities above satisfy the relation

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \min\{\mathcal{G}_E(\bar{x}_0), \mathcal{G}_E(x_0), \mathcal{G}_E(x_0) \\ + \mathcal{G}_E(a(t;\cdot))\} =: L_0 > 0, \quad t \in Co(\mathbb{R}^+). \tag{6.10}$$

The growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L_0, \quad t \in Co(\mathbb{R}^+). \tag{6.11}$$

b) Assume that (6.8) *holds and* $z(t), t \geq 0$ *is a constant,* $z(t) = \eta$, *say. If the condition*

$$\min_{j=1}^{5}\mathcal{G}_E(\Phi_j) - 1 + \min\{\mathcal{G}_E(\bar{x}_0), \ \mathcal{G}_E(x_0), \ \mathcal{G}_E(x_0) \\ + \mathcal{G}_E(a(t;\cdot)), \ \mathcal{G}_E(a(t;\cdot)) + \mathcal{G}_E(\bar{x}_0), \\ \mathcal{G}_E(x_0) + 2\mathcal{G}_E(a(t;\cdot))\} =: L_1 > 0, \quad t \in Co(\mathbb{R}^+) \tag{6.12}$$

holds, then we have

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co(\mathbb{R}^+), \tag{6.13}$$

and

$$\mathcal{G}_E\big(\frac{d}{dt}\mathcal{E}(t;\cdot)\big) \geq L_1, \;\; t \in Co(\mathbb{R}^+). \tag{6.14}$$

Proof. We start with the proof of (6.9). Due to (6.10), given any small $\varepsilon > 0$ and $N \in (0, L_0 - \varepsilon)$ we take reals $\zeta > 0$ and τ, σ, ν near to $-\mathcal{G}_E(\hat{x}_0)$, $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(a(t;\cdot))$ respectively, such that

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N + \max\{\tau, \sigma, \sigma+\nu\}.$$

Hence (4.34) and (4.35) keep in force. These arguments and Lemma 2.2 imply that (4.32) hold, for some $K > 0$ and $q := E(p)$ with $p \geq p_0 \geq 1$. Notice, also, that

$$\max\{\tau, \;\; \sigma, \;\; \sigma+\nu\} - \zeta < -N. \tag{6.15}$$

Because of (6.15) we can obtain some $\delta > 0$ and $p_1 \geq p_0$ such that

$$\frac{5\delta}{2} + \frac{K}{2}q^{-\zeta} + \max\{\tau, \;\; \sigma+\delta, \;\; \sigma+\nu\} - \zeta < -N, \;\; p \geq p_1. \tag{6.16}$$

Keep in mind assumption (i) of the theorem, relations (4.34) and (4.35), for some positive constants K_2, K_3, K_4 and, moreover,

$$b(t;p) \leq -\theta, \tag{6.17}$$

for all t and p large. Fix any $\hat{t} > 0$ and define

$$\lambda := \int_0^{\hat{t}} \sqrt{z(s)ds}.$$

Obviously there is a $p_2 \geq p_1$ such that for all $q \geq p_2$, we have

$$Kq^{-\zeta} \leq 1, \;\; q \geq p_2 \tag{6.18}$$

and

$$\log(\log(u)) \leq \log(u) \leq u^{\delta}, \;\; u \geq p_2. \tag{6.19}$$

Consider the function $\mathcal{L}(t;p)$ defined in (6.5). Then due to (4.32), (3.14), (4.33), (4.34), (6.17), (6.8) and (6.19), for all $t \in [0, \hat{t}]$ and

$q \geq p_2$, we have

$$\begin{aligned}\mathcal{L}(t;p) &\leq P(p)\kappa(p)\Big(\frac{z(0)}{\theta}\Big)^{\frac{1}{4}}\mathcal{G}_E\Big[\log(\log(q))\Big]^{\frac{3}{2}}\sinh[\mathcal{G}_E\log(\log(q))]\\ &\quad\times\exp\big[P(p)\cosh[\log(\log(q))]\big]\\ &\leq \mathcal{G}_E P(p)\kappa(p)\Big(\frac{z(0)}{\theta}\Big)^{\frac{1}{4}}q^{\frac{3\delta}{2}}\frac{1}{2}q^{\delta}p^{\frac{P(p)}{2}}\exp(\frac{\mathcal{G}_E P(p)}{2\log(q)})\\ &\leq \mathcal{G}_E K q^{-\zeta}\frac{1}{2}\Big[K_3q^{\sigma}+\frac{K_2}{\sqrt{\theta}}q^{\tau}+\frac{K_3}{\sqrt{\theta}}q^{\sigma}\Big(\frac{\sqrt{K}}{4}q^{-\frac{\zeta}{2}}q^{\delta}\sqrt{z(0)}+\frac{K_4}{2}q^{\nu}\Big)\Big]\\ &\quad\times\Big(\frac{z(0)}{\theta}\Big)^{\frac{1}{4}}q^{\frac{3\delta}{2}}\frac{1}{2}q^{\delta}q^{\frac{P(p)}{2}}\exp(\frac{\mathcal{G}_E P(p)}{2\log(p)})e^{\frac{1}{\lambda}}.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}\mathcal{L}(t;p) &\leq \Lambda_1 q^{-\zeta+\sigma+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}+\Lambda_2 q^{-\zeta+\tau+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}\\ &\quad+\Lambda_3 q^{-\zeta+\sigma-\frac{\zeta}{2}+\delta+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}+\Lambda_4 q^{-\zeta+\sigma+\nu+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}},\end{aligned}\tag{6.20}$$

for some constants $\Lambda_j, j=1,2,3,4$. From (6.16) and (6.20) we obtain

$$\mathcal{L}(t;p) \leq \Lambda_0 q^{-N}, \quad t\in[0,\hat{t}]\tag{6.21}$$

for some $\Lambda_0>0$. This and (6.5) complete the proof of (6.9).

Now, from the previous arguments it follows that given any $L\in(0,N)$ it holds

$$\mathcal{L}(t;p)q^L \leq \Lambda_0 q^{-N+L}\to 0, \text{ as } p\to+\infty,$$

where, notice that, the constant Λ_0 is uniformly chosen for t in the interval $[0,\hat{t}]$ and p with $E(p)\geq p_2$. This gives

$$\mathcal{E}(t;p)q^L\to 0, \text{ as } p\to+\infty, \ t\in Co(\mathbb{R}^+).$$

Hence the growth index of the error function $\mathcal{E}$ satisfies $\mathcal{G}_E(\mathcal{E}(t;p))\geq L$ and so we get

$$\mathcal{G}_E(\mathcal{E}(t;p))\geq N \text{ as } p\to+\infty.$$

Since N is arbitrary in the interval $(0,N_0-\varepsilon)$ and ε is small, we get (6.11).

(b) Fix any $\hat{t}>0$ and take any small $\varepsilon>0$ and $N\in(0,L_1-\varepsilon)$. Also from (6.12) we can get $\zeta>0$, $\delta>0$ and reals σ,ν,τ as above, such that

$$\max\Big\{\frac{5\delta}{2}+1 + \max\{\delta+\sigma+\nu,\ \delta+\tau, \nu+\tau,\ 2\delta+\sigma,\ 2\nu+\sigma\}, 2\delta+\hat{t}\sqrt{\eta}\delta+1+\max\{\tau,\delta+\sigma,\sigma+\nu\}\Big\}+N < \zeta < \min_{j=1}^{5}\mathcal{G}_E(\Phi_j). \tag{6.22}$$

Such a δ may be chosen in such way that

$$\hat{t}\sqrt{\eta}\delta < 1.$$

By using inequality (6.6) and relation (3.8) we get

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &\le \mathcal{L}(t;p)\Big[\frac{\sqrt{\Phi_1(p)|b(t;p)|}}{4}+\frac{|a(t;p)|}{2}\Big]\\ &+\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac14} exp\Big(-\frac12\int_0^{\hat{t}} a(s)ds\Big)\sqrt{-b(t;p)}\\ &\times\Big[\int_0^t\sqrt{-b(s;p)}ds\cosh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)\\ &+\sinh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)\\ &+e^{P(p)}\int_0^t\sqrt{-b(s;p)}ds\cosh\Big(\int_0^t\sqrt{-b(s;p)}ds\Big)\\ &\times\Big(exp\Big(P(p)(\cosh(\int_0^t\sqrt{-b(s;p)}ds)-1)\Big)-1\Big)\Big],\end{aligned}$$

namely

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &\le \mathcal{L}(t;p)\Big[\frac14 K^{\frac12}p^{\frac{-\zeta}{2}}\sqrt{\eta}\log(\log(q))+\frac{K_4q^{\nu}}{2}\Big]\\ &+\Big(\frac{\eta}{\theta}\Big)^{\frac14}\big(\log(\log(q))\big)^{\frac12}Kq^{-\zeta}\\ &\times\frac12\Big[K_3q^{\sigma}+\frac{1}{\sqrt{\eta}}\Big(K_2q^{\tau}\\ &+K_3q^{\sigma}\Big[\frac14 K^{\frac12}q^{\frac{-\zeta}{2}}\sqrt{\eta}\log(\log(q))+\frac{K_4p^{\nu}}{2}\Big]\end{aligned}$$

$$\times\Big[\hat{t}\sqrt{\eta}(\log(\log(q)))\cosh(\hat{t}\sqrt{\eta}\log(\log(q)) + \sinh(\hat{t}\sqrt{\eta}\log(\log(q))) + e^{Kq^{-\zeta}}\hat{t}\sqrt{\eta}\log(\log(q))\cosh(\hat{t}\sqrt{\eta}\log(\log(q))) \times\Big(\exp\Big(Kq^{-\zeta}(\cosh(\hat{t}\sqrt{\eta}\log(\log(q)))-1)\Big)-1\Big)\Big]\lambda q^{\frac{\delta}{2}}.$$

Letting any p with $q := E(p) \geq p_0 > e$, and p_0 being such that

$$q \geq p_0 \Longrightarrow \log(q) \leq q^{\delta}$$

and using the fact that

$$x > 0 \Longrightarrow \cosh(x) \leq e^{x} \text{ and } \sinh(x) \leq \frac{1}{2}e^{x},$$

from the previous estimate, we get

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq \Big[\Lambda_1 q^{-\zeta+\sigma+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}} + \Lambda_2 q^{-\zeta+\tau+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}} + \Lambda_3 q^{-\zeta+\sigma-\frac{\zeta}{2}+\delta+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}} + \Lambda_4 q^{-\zeta+\sigma+\nu+\frac{3\delta}{2}+\delta+\frac{K}{2}q^{-\zeta}}\Big] \times\Big[\frac{1}{4}K^{\frac{1}{2}}q^{\frac{-\zeta}{2}}\sqrt{\eta}q^{\delta}+\frac{K_4q^{\nu}}{2}\Big]+\Big(\frac{\eta}{\theta}\Big)^{\frac{1}{4}}q^{\frac{\delta}{2}}Kq^{-\zeta} \times\frac{1}{2}\Big[K_3q^{\sigma}+\frac{1}{\sqrt{\eta}}\Big(K_2q^{\tau}+K_3q^{\sigma}\Big[\frac{1}{4}K^{\frac{1}{2}}q^{\frac{-\zeta}{2}}\sqrt{\eta}q^{\delta}+\frac{K_4q^{\nu}}{2}\Big] \times\Big[\hat{t}\sqrt{\eta}q^{\delta}p^{\hat{t}\sqrt{\eta}\delta}+\frac{1}{2}q^{\hat{t}\sqrt{\eta}\delta}+e^{Kq^{-\zeta}}\hat{t}\sqrt{\eta}q^{\delta}q^{\hat{t}\sqrt{\eta}\delta} \times\Big(\exp\Big(Kq^{-\zeta}((\log(q))^{\hat{t}\sqrt{\eta}\delta})\Big)\Big]\lambda q^{\frac{\delta}{2}}.$$

Therefore it follows that

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq \sum_{j=1}^{20}\Gamma_j q^{r_j}, \tag{6.23}$$

for some positive constants Γ_j, $j = 1, 2, \cdots, 20$ and

$$r_1 := -\zeta-\frac{\zeta}{2}+\sigma+\frac{7\delta}{2}+1,\ \ r_2 := -\zeta+\sigma+\frac{5\delta}{2}+1+\nu,$$
$$r_3 := -\zeta-\frac{\zeta}{2}+\tau+\frac{7\delta}{2}+1,\ \ r_4 := -\zeta+\tau+\frac{5\delta}{2}+1+\nu,$$
$$r_5 := -2\zeta+\sigma+\frac{9\delta}{2}+1,\ \ r_6 = r_7 := -\zeta-\frac{\zeta}{2}+\sigma+\frac{7\delta}{2}+1+\nu,$$
$$r_8 := -\zeta+\sigma+\frac{5\delta}{2}+1+2\nu,\ \ r_9 := 2\delta-\zeta+\sigma+\hat{t}\sqrt{\eta}\delta,$$
$$r_{10} := \delta-\zeta+\sigma+\hat{t}\sqrt{\eta}\delta,\ \ r_{11} := 2\delta-\zeta+\sigma+\hat{t}\sqrt{\eta}\delta+1,$$

$$r_{12} := 2\delta - \zeta + \tau + \hat{t}\sqrt{\eta}\delta, \quad r_{13} := \delta - \zeta + \tau + \hat{t}\sqrt{\eta}\delta,$$

$$r_{14} := 2\delta - \zeta + \tau + \hat{t}\sqrt{\eta}\delta + 1, \quad r_{15} := 3\delta - \zeta - \frac{\zeta}{2} + \sigma + \hat{t}\sqrt{\eta}\delta,$$

$$r_{16} := 2\delta - \zeta + \sigma - \frac{\zeta}{2} + \hat{t}\sqrt{\eta}\delta, \quad r_{17} := 3\delta - \zeta + \sigma - \frac{\zeta}{2} + \hat{t}\sqrt{\eta}\delta + 1,$$

$$r_{18} := 2\delta - \zeta + \sigma + \nu + \hat{t}\sqrt{\eta}\delta, \quad r_{19} := \delta - \zeta + \sigma + \nu + \hat{t}\sqrt{\eta}\delta,$$

$$r_{20} := 2\delta - \zeta + \sigma + \nu + \hat{t}\sqrt{\eta}\delta + 1.$$

Due to (6.22) all the previous constants are smaller than $-N$. Then, for the quantity $\Gamma_0 := \max_j \Gamma_j$, inequality (6.23) gives

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq \Gamma_0 q^{-N}, \quad q \geq p_0, \tag{6.24}$$

which leads to (6.9), since the constant N is arbitrary.

The proof of the claim (6.11) follows from (6.24) in the same way as (4.22) follows from (4.38).

□

Theorem 6.3. *Consider the initial value problem* (1.5) - (1.6), *where the conditions of Theorem 6.1 and (i), (ii), (iii) of Theorem 6.2 keep in force. Assume, also, that* (4.41) *and* (6.8) *hold.*

a) If relation (6.12) *is true, then*

$$\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co([0, T(L_0))).$$

Moreover the growth index at infinity of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L_0, \quad t \in Co([0, T(L_0))).$$

b) If (6.12) *keeps in force, then*

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co([0, T(L_1)))$$

and

$$\mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;p)) \geq L_1, \quad t \in Co([0, T(L_1))).$$

Proof. First of all we can see that for a fixed $\hat{t} \in (0, T(L_0))$, due to (4.41) and (4.44) we can find reals τ, σ, ν near to $-\mathcal{G}_E(\hat{x}_0)$, $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(a(t;\cdot))$, respectively, such that

$$\exp\Big(-\frac{1}{2}\int_0^{\hat{t}} a(s;p)ds\Big) \leq p^{\frac{1}{2}\Omega(\hat{t})}.$$

Taking into account this fact and relation (6.10), we can see that

$$\max\{\tau, \sigma, \sigma+\nu\} + \frac{1}{2}\Omega(\hat{t}) < \min_{j=1}^{5} \mathcal{G}_E(\Phi_j).$$

Now, we proceed as in the proof of Theorem 6.2, where it is enough to observe that the right hand side of relation (6.20) is multiplied by the factor

$$\exp(-\frac{1}{2}\int_0^{\hat{t}} a(s;p)ds).$$

A similar procedure is followed for the proof of part (b) of the theorem. □

7. A specific case of the Initial Value Problem (1.3)-(1.4)

We shall apply the results of theorem 6.2 to a specific case of the problem (1.3) - (1.4), namely to the problem

$$x'' + 2ap^{\nu}x' - a^2p^{2\mu}x + p^m x\sin(x) = 0, \tag{7.1}$$

associated with the initial conditions

$$x(0;p) = ap^{\sigma}, \quad x'(0;p) = ap^{\tau}, \tag{7.2}$$

where, for simplicity, we have set

$$a := \frac{1}{10}, \quad \mu := 2, \quad \nu := \frac{1}{9}, \quad \tau = \sigma := \frac{1}{2}, \quad m \leq \frac{2}{9}.$$

Using these quantities we can see that all assumptions of Theorem 6.2 hold, with $E(p) = p$,

$$L_0 = \frac{19}{6}, \quad L_1 = \frac{7}{6}.$$

Then an approximate solution of the problem is given by

$$\tilde{x}t;p) := \frac{1}{10}e^{-\frac{t}{10}p^{\frac{1}{9}}}p^{\frac{1}{2}}\cosh(\frac{p^2t}{10}) + (10p^{-\frac{3}{2}} + p^{\frac{11}{18}})\sinh(\frac{p^2t}{10}), \quad t \geq 0.$$

In Figure 2 the approximate solution for the values p=1, 3.45, 5.90, 8.38, 10.80, 13.25, 15.70, 18.15 is shown.

8. Approximate solutions of the Boundary Value Problem (1.9)-(1.10)

In this section we consider Eq. (1.9) associated with the boundary conditions (1.10). Our purpose is to use the results of section 3 in order to approximate the solutions of the boundary value problem, when the parameter p approaches the critical value $+\infty$.

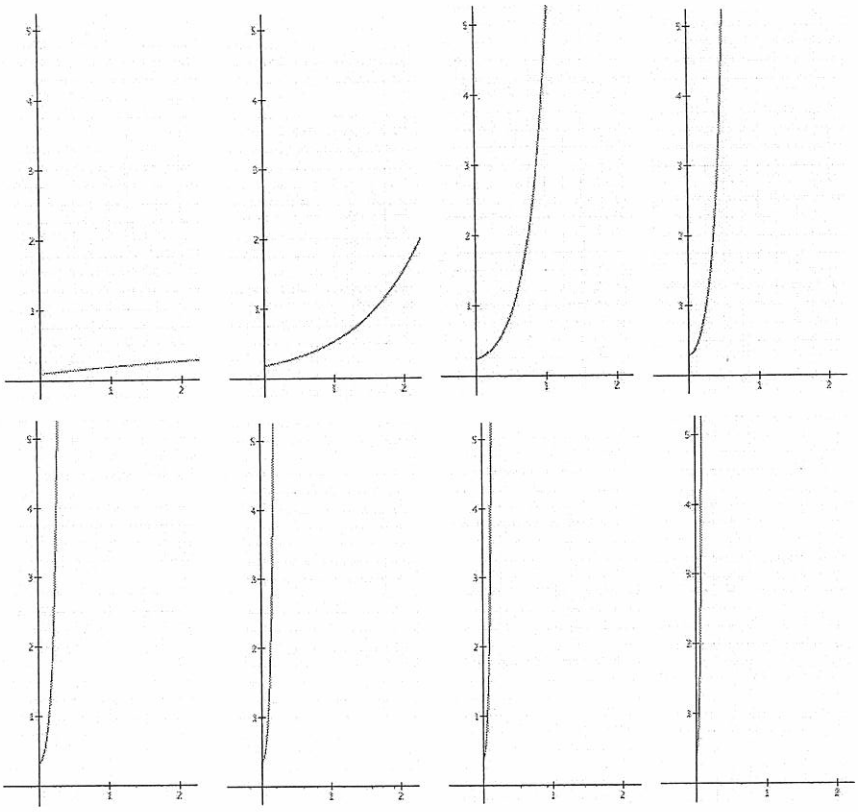

FIGURE 2. Approximate solutions of (7.1) - (7.2), when p=1, 3.45, 5.90, 8.38, 10.80, 13.25, 15.70, 18.15 respectively

To begin with define $\tau := v(1;p)$ and from now on the letter J_p will denote the interval $[0,\tau]$. Also, in order to unify our results, we make the following convention:

We shall denote by

$$S_c(v) = \begin{cases} \sin(v), & if\ \ c = +1 \\ \sinh(v), & if\ \ c = -1, \end{cases}$$

$$C_c(v) = \begin{cases} \cos(v), & if\ \ c = +1 \\ \cos(v), & if\ \ c = -1. \end{cases}$$

Our basic hypothesis which will be assumed in all the sequel without any mention is the following:

Condition 8.1. *In case $c = +1$ let*

$$\tau := v(1;p) = \int_0^1 \sqrt{b(s;p)}ds < \pi, \tag{8.1}$$

for all p large enough.

Suppose that the problem (1.9)-(1.10) admits a solution $x(t;p)$, $t \in [0,1]$. Then, Theorem 3.2 implies, and inversely, that if $y(\cdot;p)$ is a solution of equation (3.10) having boundary conditions

$$\begin{aligned} y(0;p) &= x_0(p) =: y_0(p) \\ y(\tau;p) &= y(v(1;p);p) = \frac{x(1;p)}{Y(1;p)} \\ &= x_1(p)\Big(\frac{b(1;p)}{b(0;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} =: y_\tau(p). \end{aligned} \tag{8.2}$$

Before we seek for approximate solutions of the problem (1.9)-(1.10) we shall give conditions for the existence of solutions. To do that we need the following classical fixed point theorem:

Theorem 8.2. *(Nonlinear alternative)* [6]. *Let D be a convex subset of a Banach space X, let U be an open subset of D, and let $A : \bar{U} \to D$ be a completely continuous mapping. If $q \in U$ is a fixed element, then either A has a fixed point in $\bar{U}$, or there is a point $u \in \partial U$ and $\lambda \in (0,1)$, such that $u = \lambda Au + (1-\lambda)q$.*

To proceed we shall formulate the integral form of the problem and then we shall apply Theorem 8.2. To this end we let w be the solution of the homogeneous equation

$$w'' + cw = 0,$$

with boundary conditions $w(0;p) = y_0(p)$ and $w(\tau;p) = y_\tau(p)$. This means that w is defined as

$$w(v;p) = \frac{1}{S_c(\tau)}\big(y_0(p)(S_c(\tau - v) + y_\tau(p)S_c(v)\big). \tag{8.3}$$

(Notice that because of (8.1) in case $c = +1$ the factor $S_c(\tau)$ is positive for all τ.) Hence we see that

$$|w(v;p)| \leq q_c(|y_0| + |y_\tau|),$$

where

$$q_c := \begin{cases} \frac{1}{\min\{\sin(\sqrt{\theta}),\sin(\tau)\}}, & c = +1 \\ \frac{\sinh(\tau)}{\sinh(\sqrt{\theta})}, & c = +1. \end{cases}$$

Next we let $R(v;p),\ \ v \in J$ be the solution of equation

$$R''(v;p) + cR(v;p) = H(v;p),\ \ v \in J_p \tag{8.4}$$

satisfying the boundary conditions

$$R(0;p) = R(\tau;p) = 0. \tag{8.5}$$

where

$$\begin{aligned} H(v;p) &:= C(t, y(v;p);p)y(v;p) \\ &= C(t, y(v;p);p)R(v;p) + C(t, y(v;p);p)w(v;p). \end{aligned}$$

The latter, due to (3.15), implies that

$$|H(v;p)| \le P(p)|R(v;p)| + P(p)q_c(|y_0(p)| + |y_\tau(p)|). \tag{8.6}$$

To formulate an integral form of the problem we follow an elementary method and obtain

$$R(v;p) = d_1C_c(v) + d_2S_c(v) + \int_0^v S_c(v-s)H(s;p)ds,\ \ v \in J_p \tag{8.7}$$

for some constants d_1, d_2 to be determined from the boundary values (8.5). Thus we have

$$0 = R(0;p) = d_1$$

and

$$0 = R(\tau;p) = d_1C_s(\tau) + d_2S_c(\tau) + \int_0^\tau S_c(\tau - s)H(s;p)ds.$$

This implies that

$$d_2 = -\frac{1}{S_c(\tau)}\int_0^\tau S_c(\tau - s)H(s;p)ds$$

and so we have

$$R(v;p) = \int_0^\tau G(v,s;p)H(s;p)ds, \tag{8.8}$$

where the one-parameter Green's function G is defined by

$$G(v,s;p) := \frac{-S_c(v)S_c(\tau - s)}{S_c(\tau)} + S_c(v-s)\chi_{[0,v]}(s). \tag{8.9}$$

Here the symbol χ_A denotes the characteristic function of the set A. From (8.9) we can see that

$$G(v,s;p) = \begin{cases} -\frac{S_c(s)S_c(\tau - v)}{S_c(\tau)}, & 0 \le s \le v \le \tau \\ -\frac{S_c(v)S_c(\tau - s)}{S_c(\tau)}, & 0 \le v \le s \le \tau \end{cases}$$

From 3.1 and (8.1) it follows that for all $s, v \in [0, \tau]$ it holds

$$\max\{|G(v,s;p)|,\ |\frac{\partial}{\partial v}G(v,s;p)|\} \le Q_c, \tag{8.10}$$

where

$$Q_c := \begin{cases} \frac{1}{\min\{\sin(\sqrt{\theta}),\sin(\tau)\}}, & c = +1 \\ \frac{(\sinh(\tau))^2}{\sinh(\sqrt{\theta})}, & c = -1. \end{cases}$$

Now we see that the operator form of the boundary value problem (3.10)-(8.2) is the following:

$$y(v;p) = w(v;p) + \int_0^{\tau} G(v,s;p)C(\phi(s;p), y(s;p);p)y(s;p)ds, \quad v \in J_p. \tag{8.11}$$

To show the existence of a solution of (8.11) we consider the space $C(J_p, \mathbb{R})$ of all continuous functions $y : J_p \to \mathbb{R}$ endowed with the sup-norm $\|\cdot\|$-topology. This is a Banach space. Fix a p large enough and define the operator $A : C(J_p, \mathbb{R}) \to C(J_p, \mathbb{R})$ by

$$(Az)(v) := w(v;p) + \int_0^{\tau} G(v,s;p)C(\phi(s;p), z(s);p)z(s)ds$$

which is completely continuous (due to Properties 3.1 and 3.3).

To proceed we assume for a moment that it holds

$$1 - P(p)\tau Q_c =: \Delta(p) = \Delta > 0, \tag{8.12}$$

where (recall that) $P(p)$ is defined in (3.15). Take any large p and let $\tau = v(1;p) =: v$. Then, clearly,

$$1 - P(p)\tau Q_c \geq \Delta > 0$$

Consider the open ball $B(0,l)$ in the space $C(J_p, \mathbb{R})$, where

$$l := \frac{\|w\|}{1 - P(p)\tau Q} + 1.$$

Here $\|w\|$ is the sup-norm of w on J_p.

Assume that the operator A does not have any fixed point in $B(0,l)$. Thus, due to Theorem 8.2 and by setting $q = 0$, there exists a point z in the boundary of $B(0,l)$ satisfying

$$z = \lambda Az,$$

for some $\lambda \in (0,1)$. This means that for each $v \in J_p$ it holds

$$|z(v)| \leq \|w\| + \int_0^{\tau} |G(v,s;p)||C(\phi(s;p), z(s);p)||z(s)|ds.$$

Then, from (8.10) we have

$$|z(v)| \leq \|w\| + Q_c P(p) \int_0^{\tau} |z(s)|ds.$$

Thus, we get

$$|z(v)| \leq \|w\| + Q_c P(p)\tau\|z\|, \tag{8.13}$$

which leads to the contradiction

$$l = \|z\| \leq \frac{\|w\|}{1 - P(p)\tau Q_c} = l - 1.$$

Taking into account the relation between the solutions of the original problem and the solution of the problem (1.9)-(1.10), as well the previous arguments, we conclude the following result:

Theorem 8.3. *If Properties 3.1, 3.3 and* (8.12) *are true, then the boundary value problem* (1.9)-(1.10) *admits at least one solution.*

Now, we give the main results of this section. First we define the function

$$\begin{aligned}
\tilde{x}(t;p) :=& \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big)\frac{1}{S_c(\int_0^1 \sqrt{b(s;p)}ds)} \\
&\times \Big\{x_0(p)S_c(\int_t^1 \sqrt{b(s;p)}ds) \\
&+ x_1(p)\Big(\frac{b(1;p)}{b(0;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} S_c(\int_0^t \sqrt{b(s;p)}ds)\Big)\Big\} \\
=& \frac{1}{S_c(\int_0^1 \sqrt{b(s;p)}ds)}\Big\{\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \\
&\times \exp\big(-\frac{1}{2}\int_0^t a(s;p)ds\big)S_c(\int_t^1 \sqrt{b(s;p)}ds)x_0(p) \\
&+ \Big(\frac{b(1;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} S_c(\int_0^t \sqrt{b(s;p)}ds))x_1(p)\Big\}
\end{aligned} \tag{8.14}$$

which is going to be an approximate solution of the problem.

Theorem 8.4. *Consider the boundary value problem* (1.9) - (1.10), *where assume that Properties 3.1, 3.3, 8.1 , the conditions (i), (ii) of Theorem 4.2 and assumption* (4.41) *keep in force. Also, assume that the boundary values have a behavior like*

$$x_0, \in \mathcal{A}_E. \tag{8.15}$$

a) If the condition

$$\begin{aligned}
&\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \frac{3}{4}\mathcal{G}_E(b(t;\cdot)) - \Omega + \min\{\mathcal{G}_E(x_0), \mathcal{G}_E(x_1)\} \\
&=: L_0 > 0
\end{aligned} \tag{8.16}$$

is satisfied, then the existence of a solution x of the problem is guaranteed and if

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p) \tag{8.17}$$

is the error function, where $\tilde{x}$ is defined by (8.14), *then we have*

$$\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co([0,1]), \tag{8.18}$$

where

$$\Omega := \frac{1}{2}\int_0^1 \omega(s)ds.$$

(Here ω is given in assumption (4.41)*.)*

Also, the growth index of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L_0, \quad t \in Co([0,1]). \tag{8.19}$$

b) Assume that the condition

$$\begin{aligned}\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \frac{3}{4}\mathcal{G}_E(b(t;\cdot)) - \Omega + \min\{&\mathcal{G}_E(x_0) + \frac{1}{2}\mathcal{G}_E(b(t;\cdot)),\\ &\mathcal{G}_E(x_0) + \mathcal{G}_E(b(t;\cdot)), \mathcal{G}_E(x_1) + \frac{1}{2}\mathcal{G}_E(b(t;\cdot)), \mathcal{G}_E(x_0),\\ &\mathcal{G}_E(x_1)\} =: L_1, \quad t \in Co([0,1]) > 0,\end{aligned} \tag{8.20}$$

holds. Then the existence of a solution x of the problem is guaranteed and it satisfies

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \quad p \to +\infty, \quad t \in Co([0,1]), \tag{8.21}$$

and

$$\mathcal{G}_E\Big(\frac{d}{dt}\mathcal{E}(t;\cdot)\Big) \geq L_1, \quad t \in Co([0,1]). \tag{8.22}$$

Proof. a) Take any $N \in (0, L_0)$ and, because of (8.16), we can choose $\zeta > 0$ and real numbers μ, σ, ϱ near to $-\mathcal{G}_E(b(t;\cdot))$, $-\mathcal{G}_E(x_0)$, $-\mathcal{G}_E(x_1)$, respectively, such that

$$\min_{j=1}^{5} \lambda(\Phi_j) > \zeta \geq N + \frac{\mu}{4} + \Omega + \max\{\sigma, \varrho\}. \tag{8.23}$$

Thus, we have

$$\frac{\mu}{4} + \Omega + \max\{\sigma, \varrho\} - \zeta \leq -N \tag{8.24}$$

and, and due to Lemma 2.2,

$$P(p) \leq K(E(p))^{-\zeta}, \tag{8.25}$$

for some $K > 0$. Thus (8.12) keeps in force for p large enough. This makes Theorem 8.3 applicable and the existence of a solution is guaranteed.

Let $\mathcal{E}(t;p)$ be the error function defined in (8.17). From (8.8), (8.10) and (8.6) we have

$$|R(v;p)| \leq q_c Q_c P(p)\tau(|y_0| + |y_\tau|) + Q_c P(p) \int_0^\tau |R(s;p)| ds,$$

and therefore

$$\begin{aligned} |R(v;p)| &\leq \frac{q_c Q_c P(p)\tau(|y_0| + |y_\tau|)}{1 - Q_c P(p)\tau} \\ &\leq \frac{1}{\Delta} q_c Q_c P(p)\tau(|y_0| + |y_\tau|), \quad v \in J_p. \end{aligned} \tag{8.26}$$

Then observe that

$$\begin{aligned} |\mathcal{E}(t;p)| &= |x(t;p) - Y(t;p)w(v(t;p);p)| \\ &= |Y(t;p)||y(v(t;p);p) - w(v(t;p);p)| = |Y(t;p)||R(v(t;p);p)|, \end{aligned}$$

because of (4.4). Thus, from (8.26) it follows that for all $t \in [0,1]$ it holds

$$\begin{aligned} |\mathcal{E}(t;p)| &\leq \Delta^{-1}|Y(t;p)| q_c Q_c P(p)\tau(|y_0| + |y_\tau|) \\ &= \Delta^{-1}\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{-\frac{1}{2}\int_0^t a(s;p)ds} q_c Q_c P(p)\tau\big(|y_0| + |y_\tau|\big) \\ &= \Delta^{-1} q_c Q_c \sqrt{\|b(\cdot;p)\|} P(p) \\ &\times \Big[\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{-\frac{1}{2}\int_0^t a(s;p)ds}|x_0(p)| \\ &+ \Big(\frac{b(1;p)}{b(t;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_t^1 a(s;p)ds}|x_1(p)|\Big]. \end{aligned} \tag{8.27}$$

From (8.25) and (8.27) for all large p (especially for all p with $q := E(p) > 1$) it follows that

$$\begin{aligned} |\mathcal{E}(t;p)| &\leq \Delta^{-1} q_c Q_c \tau K q^{-\zeta} \\ &\times \frac{K_1^{\frac{3}{4}} q^{\frac{3\mu}{4}}}{\theta^{\frac{1}{4}}} \exp\Big(\log(q)\frac{1}{2}\int_0^1 \omega(s)ds\Big)\Big(K_2 q^\sigma + K_3 q^\varrho\Big) \\ &\leq K_4 q^{-\zeta + \frac{3\mu}{4} + \Omega}(K_2 q^\sigma + K_3 q^\varrho). \end{aligned}$$

Finally, from (8.24) we get

$$|\mathcal{E}(t;p)| \leq \hat{K} q^{-N}, \tag{8.28}$$

for some $\hat{K} > 0$, which, obviously, leads to (8.18). Relation (8.19) follows from (8.28) as exactly relation (4.22) follows from (4.38).

b) Next consider the first order derivative of the error function $\mathcal{E}(t;p)$. Due to (8.20), given any small ε and $N \in (0, L_1 - \varepsilon)$, we get reals $\zeta >$

0 and real $\mu, \nu, \sigma, \varrho > 0$, near to $-\mathcal{G}_E(b(t;\cdot))$, $-\mathcal{G}_E(a(t;\cdot))$, $-\mathcal{G}_E(x_0)$, $\mathcal{G}_E(x_1)$, respectively, such that

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) > \zeta > N_1 + \frac{3\mu}{4} + \Omega + \max\{\sigma + \frac{\mu}{2}, \sigma + \nu, \varrho + \frac{\mu}{2}, \varrho + \nu, \mu + \varrho, \mu + \sigma\}. \tag{8.29}$$

From (8.9) and (8.10) we observe that it holds

$$\begin{aligned}|\frac{d}{dv}R(v;p)| &= |\frac{d}{dv}\int_0^{\tau} G(v,s;p)H(s;p)ds| \\ &\leq Q_c\tau(P(p)|R(v;p)| + P(p)q_c(|y_0| + |y_{\tau|}) \\ &\leq q_cQ_c\tau P(p)[\Delta^{-1}Q_c\tau P(p) + 1](|y_0| + |y_\tau|).\end{aligned}$$

From this relation it follows that

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &= |\frac{d}{dt}Y(t;p)R(v(t;p);p) + Y(t;p)\frac{d}{dv}R(v(t;p);p)\frac{d}{dt}v(t;p)| \\ &\leq |Y(t;p)|\Big\{\Big(\frac{\sqrt{\Phi_1(p)b(t;p)}}{4} + \frac{|a(t;p)|}{2}\Big)|R(v(t;p);p)| \\ &\quad + |\frac{d}{dv}R(v(t;p);p)|\sqrt{b(t;p)}\Big\} \\ &\leq |Y(t;p)|\Big\{\Big(\frac{\sqrt{\Phi_1(p)b(t;p)}}{4} + \frac{|a(t;p)|}{2}\Big) \\ &\quad \times \Delta^{-1}q_cQ_cP(p)\tau(|y_0| + |y_\tau|) \\ &\quad + \sqrt{b(t;p)}q_cQ_c\tau P(p)[\Delta^{-1}Q_c\tau P(p) + 1](|y_0| + |y_\tau|)\Big\}.\end{aligned}$$

Therefore, for all large p (especially for p with $q := E(p) > 1$) we obtain

$$\begin{aligned}|\frac{d}{dt}\mathcal{E}(t;p)| &\leq q_cQ_c\hat{\tau}P(p)\Big[|x_0(p)|\Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}}e^{-\int_0^t a(s;p)ds} \\ &\quad + |x_1(p)|\Big(\frac{b(1;p)}{b(t;p)}\Big)^{\frac{1}{4}}e^{\int_t^1 a(s;p)ds}\Big] \\ &\quad \times \Big\{\Big(\frac{\sqrt{\Phi_1(p)b(t;p)}}{4} + \frac{|a(t;p)|}{2}\Big)\Delta^{-1} \\ &\quad + \sqrt{b(t;p)}[\Delta^{-1}Q_c\tau P(p) + 1]\Big\} \\ &\leq q^{-\zeta+\Omega+\frac{3\mu}{4}}(M_1q^{\sigma+\frac{\mu}{2}} + M_2q^{\sigma+\nu} \\ &\quad + M_3q^{\rho+\frac{\mu}{2}} + M_4q^{\rho+\nu} + M_5q^{\varrho+\mu} + M_6q^{\varrho+\sigma}),\end{aligned} \tag{8.30}$$

for some positive constants $M_1, M_2, M_3, M_4, M_5, M_6$ not depending on the parameter p. Taking into account the condition (8.29) we conclude that

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq Mq^{-N_1},$$

for all large p. Now, the rest of the proof follows as previously. □

From inequalities (8.27) and (8.30) we can easily see that if the function $a(\cdot;p)$ is non-negative uniformly for all p and $x_1(p) = 0$, or $a(\cdot;p)$ is non-positive uniformly for all p and $x_0(p) = 0$, then the conditions of Theorem 8.4 can be weaken. Indeed, we have the following results, whose the proofs follow the same lines as in Theorem 8.4:

Theorem 8.5. *Consider the boundary value problem* (1.9) - (1.10), *where assume that Properties 3.1, 3.3, 8.1 and the conditions (i), (ii) of Theorem 4.2 hold.*

Also, assume that $a(t;p) \geq 0$ *[respectively* $a(t;p) \leq 0$,*] for all* $t \in [0,1]$ *and* p *large, as well as*

$$x_0 \in \mathcal{A}_E \ \text{ and } \ x_1(p) = 0, \ \text{ for all large } p$$

[resp.

$$x_0(p) = 0, \ \text{ for all large } p \ \text{ and } \ x_1(p) \in \mathcal{A}_E].$$

a) If the condition

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_0) =: L_0 > 0$$

[resp.

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j)\frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_1) = L_0 > 0]$$

is satisfied, then the existence of a solution x *of the problem is guaranteed and if*

$$\mathcal{E}(t;p) = x(t;p) - \tilde{x}(t;p)$$

is the error function, where $\tilde{x}$ *is defined by* (8.14), *then* (8.18) *holds.*

Also, the growth index at infinity of the error function satisfies (8.19).

b) If the condition

$$\min_{j=1}^{5} \lambda(\Phi_j) + \frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_0) + \min\{\frac{1}{2}\mathcal{G}_E(b(t;\cdot)), \mathcal{G}_E(a(t;\cdot))\} =: L_1 > 0 \tag{8.31}$$

[resp.

$$\min_{j=1}^{5} \lambda(\Phi_j) + \frac{1}{4}\mathcal{G}_E(b(t;\cdot)) + \mathcal{G}_E(x_1) + \min\{\frac{1}{2}\mathcal{G}_E(b(t;\cdot), \mathcal{G}_E(a(t;\cdot))\} =: L_1 > 0]$$

holds, then the existence of a solution x of the problem is guaranteed and it satisfies (8.21) *and* (8.22).

9. Applications

1. Consider the equation

$$x'' + \frac{2}{\sin(1)}\cos(t)\log(p)x' - [1+p^{10}]x + p^{-1}x\sin(x) = 0, \tag{9.1}$$

associated with boundary values

$$x_0(p) = \frac{1}{5}p, \quad x_1(p) = \frac{1}{10}(1+\frac{1}{p}). \tag{9.2}$$

Conditions (3.11), (3.12) and (3.13) are satisfied, if we get the functions

$$\Phi_1(p) = \Phi_2(p) = \Phi_3(p) = \Phi_4(p) = k_1 p^{-\frac{39}{4}}$$

and

$$\Phi_5(p) := k_2 p^{-10},$$

for some $k_1, k_2 > 0$. So case (a) of Theorem 8.4 is applicable with $E(p) := p$. It is not hard to see that an approximate solution of the problem is the function

$$\tilde{x}(t;p) := e^{-\frac{\sin(t)}{\sin(1)}}\Big[p\frac{\sinh\left((1-t)\sqrt{1+p^{10}}\right)}{\sinh\left(\sqrt{1+p^{10}}\right)} + e(p+\frac{1}{p})\frac{\sinh\left(t\sqrt{1+p^{10}}\right)}{\sinh\left(\sqrt{1+p^{10}}\right)}\Big],$$

satisfying

$$\mathcal{G}_E\big(x(t;\cdot) - \tilde{x}(t;\cdot)\big) \geq \frac{1}{4}.$$

The function for the values of $p = 1, 1.5, 2, 2.5$ has a graph shown in Figure 3.

2. Consider the equation

$$x'' + \frac{2}{\sqrt{p}}x' + [\frac{\pi}{4} + p^{-0.1}]x + \frac{x\sin(x)}{p} = 0, \tag{9.3}$$

associated with boundary values

$$x_0(p) = 0.2\sqrt{p}, \quad x_1(p) = 0. \tag{9.4}$$

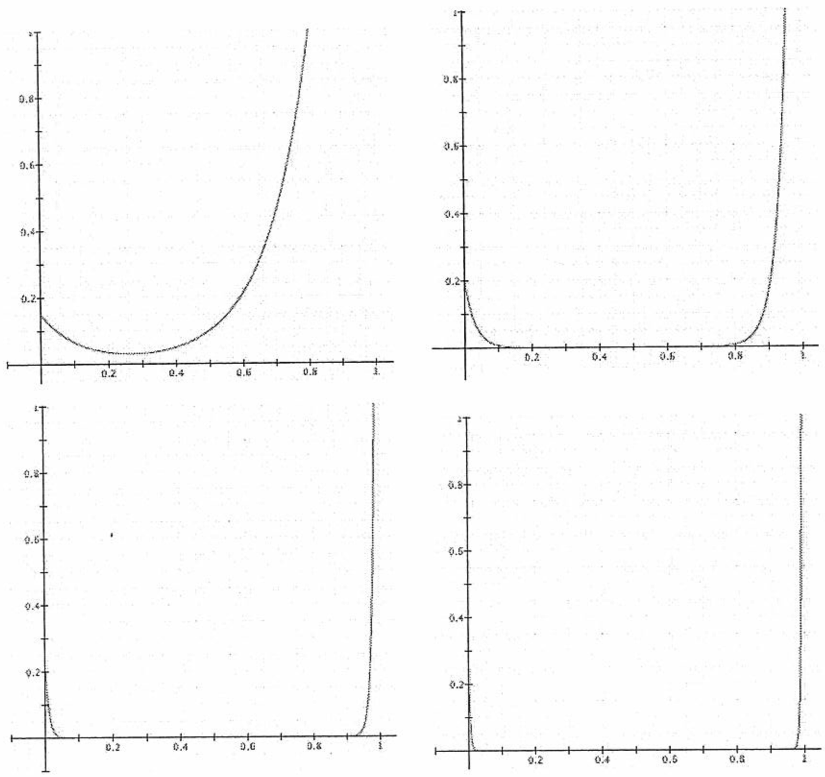

FIGURE 3. Approximate solutions of (9.1) - (9.2), when p=1, 1.5, 2, 2.5, respectively

We can take $E(p) := p$ and

$$\Phi_1(p) = \Phi_2(p) = \Phi_3(p) = \Phi_4(p) = \Phi_5(p) := k_1 p^{-0.9}.$$

Then conditions (3.11), (3.12) and (3.13) are satisfied and so Theorem 8.4 is applicable with $L_0 = \frac{3}{8}$ and $L_1 = \frac{23}{40}$.. In this case it is not hard to see that an approximate solution of the problem is the function defined on the interval $[0, 1]$ by the type

$$\tilde{x}(t;p) := 0.1\sqrt{p}(1 + \cos(15\sqrt{t}) \exp\left(\frac{-t}{\sqrt{p}}\right) \frac{\sin\left((1-t)\sqrt{\frac{\pi}{4} + p^{-0.1}}\right)}{\sin\left(\sqrt{\frac{\pi}{4} + p^{-0.1}}\right)}.$$

The graph of this function for the values of $p = 4, 10, 20, 30$ is shown in Figure 4

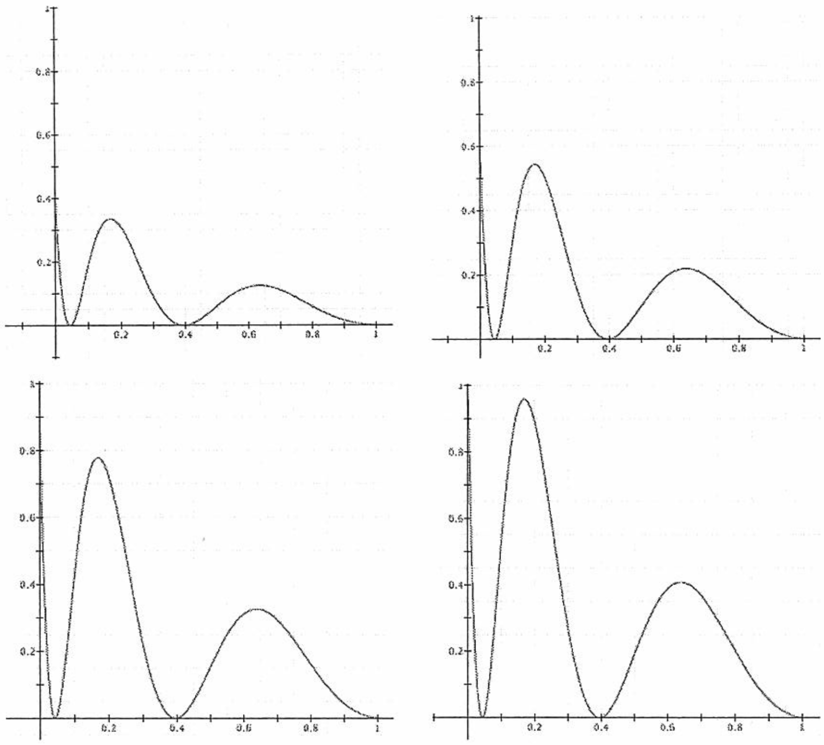

FIGURE 4. Approximate solutions of (9.3) - (9.4), when p=4, 10, 20, 30 respectively

10. APPROXIMATE SOLUTIONS OF THE BOUNDARY VALUE PROBLEM (1.9)-(1.8)

In this section we shall discuss the approximate solutions of the problem (1.9) - (1.8). We shall use the results of section 3 to obtain approximate solutions when the parameter p tends to $+\infty$. Again, as in section 8 we define $\tau := v(1;p)$, $J_P := [0,\tau]$ and use the symbols S_c and C_c.

Our basic hypothesis which will be assumed in all the sequel without any mention is that Properties 3.1 and 3.3 will keep in force for all $t \in [0,1]$.

Assume that equation (1.9) admits a solution satisfying the conditions

$$x(0;p) = x_0 \text{ and } x(1;p) = m(p)x(\xi;p),$$

for a certain point $\xi \in [0,1)$ and a real number $m(p)$. Then Theorem 3.2 implies that a function $x(\cdot;p)$ is a solution of the problem, if and only if $y(\cdot;p)$ is a solution of equation (3.10) and boundary conditions

$$\begin{aligned} y(0;p) &= x_0(p) =: y_0(p) \\ y(\tau;p) &= y(v(1);p) = \frac{x(1;p)}{Y(1;p)} = m(p)\frac{x(\xi;p)}{Y(1;p)} \\ &= m(p)\frac{Y(\xi;p)}{Y(1;p)}y(v(\xi;p);p) =: m^*(p)y(v(\xi;p);p). \end{aligned} \tag{10.1}$$

Before we seek for approximate solutions of the problem (1.9)-(1.8) we shall impose conditions for the existence of solutions. To do that we shall use, again, the Fixed Point Theorem 8.2. To proceed we assume the following:

Condition 10.1. *i) There is some $\rho > 0$ such that*

$$\frac{S_c(\int_0^\xi \sqrt{b(s;p)}ds}{S_c(\int_0^1 \sqrt{b(s;p)}ds} \geq \rho,$$

for all p large enough.

ii) It happens

$$\lim_{p\to+\infty} m(p) = +\infty.$$

iv) There is some $\bar{a} > 0$ such that

$$0 \leq a(t;p) \leq 2\bar{a},$$

for all $t \in [0,1]$ and p large enough.

iii) There are $\theta, b_0 > 0$ such that

$$\theta \leq b(t;p) \leq b_0$$

for all $t \in (0,1)$ and p large enough.

Before we seek for approximate solutions of the problem (3.10)-(10.1), we shall investigate the existence of solutions.

Let w solve the equation $w'' + cw = 0$ and satisfies the conditions

$$w(0;p) = y_0(p)$$

and

$$w(\tau;p) = m^*(p)w(v(\xi;p);p).$$

Solving this problem we obtain

$$w(v;p) = \frac{S_c(\tau - v) - m^* S_c(v(\xi;p) - v)}{S_c(\tau) - m^* S_c(v(\xi;p))} y_0(p). \tag{10.2}$$

We shall show that the solution w is bounded. Indeed, from (10.2) we observe that

$$|w(v;p)| \leq \frac{S_c(\tau) + m^* S_c(\tau)}{m^* S_c(v(\xi;p)) - S_c(\tau)} |y_0(p)|$$

and by using the bounds of all arguments involved we obtain

$$|w(v;p)| \leq \frac{m(p)\left(\frac{b(1;p)}{b(\xi;p)}\right)^{\frac{1}{4}} e^{\frac{1}{2}\int_0^1 a(s;p)ds} + 1}{m(p)\left(\frac{b(1;p)}{b(\xi;p)}\right)^{\frac{1}{4}} \frac{S_c(\int_0^\xi \sqrt{b(s;p)}ds)}{S_c(\int_0^1 \sqrt{b(s;p)}ds)} - 1} |y_0(p)|.$$

Hence, because of Condition 10.1, we obtain

$$|w(v;p)| \leq \frac{m(p)\sqrt{b_0}e^{\bar{a}} + (b_0\theta)^{\frac{1}{4}}}{m(p)\sqrt{\theta}\rho - (b_0\theta)^{\frac{1}{4}}} |y_0(p)| \leq \rho_0 |y_0(p)|, \tag{10.3}$$

for all large p, where

$$\rho_0 := \Big(\frac{\sqrt{b_0}e^{\bar{a}}}{\sqrt{\theta}\rho} + 1\Big).$$

As in previous sections, we set $R := y - w$. We shall search for constants d_1 and d_2 such that the function

$$R(v;p) := d_1 C_c(v) + d_2 S_c(v) + \int_0^v S_c(v-s)H(s;p)ds$$

be a solution of the nonhomogeneous equation

$$R'' + cR = H$$

satisfying the conditions

$$R(0;p) = 0 \text{ and } R(\tau;p) = y(\tau;p) - w(\tau;p) = m^* R(v(\xi;p)). \tag{10.4}$$

Here H is the function defined by

$$H(t;p) := C(t, y(v;p);p)R(v;p) + C(t, y(v;p);p)w(v;p),$$

which, due to (10.3), satisfies the inequality

$$|H(v;p)| \leq P(p)|R(v;p)| + P(p)\rho_0|y_0(p)|. \tag{10.5}$$

Then we obtain that

$$d_1 = 0$$

and

$$d_2 = \frac{1}{S_c(\tau) - m^*(p)S_c(v(\xi;p))}\Big[\int_0^{v(\xi;p)} S_c(v(\xi;p)) - s)H(s;p)ds - \int_0^\tau S_c(\tau - s)H(s;p)ds\Big].$$

Therefore the solution $R(v;p)$ takes the form

$$R(v;p) = \frac{S_c(v)}{S_c(\tau) - m^*(p)S_c(v(\xi;p))}\Big[\int_0^{v(\xi;p)} S_c(v(\xi;p)) - s)H(s;p)ds \\ - \int_0^{\tau} S_c(\tau - s)H(s;p)ds\Big] + \int_0^{v} S_c(v-s)H(s;p)ds,$$

namely

$$R(v;p) = \int_0^{\tau} G(v,s;p)H(s;p)ds,$$

where the Green's function G is defined by

$$G(v,s;p) := \begin{cases} \frac{S_c(v)\left[S_c(v_\xi - s) - S_c(\tau - s)\right]}{S_c(\tau) - m^*(p)S_c(v(\xi;p))} + S_c(v-s), \\ \qquad\qquad 0 \le s \le v_\xi < v \\ -\frac{S_c(v)S_c(\tau-s)}{S_c(\tau)-m^*(p)S_c(v(\xi;p))} + S_c(v-s), \\ \qquad\qquad 0 \le v_\xi < s < v \\ -\frac{S_c(v)S_c(\tau-s)}{S_c(\tau)-m^*(p)S_c(v(\xi;p))}, \qquad 0 \le v_\xi < v < s. \end{cases}$$

To obtain upper C^1 bounds of the kernel G we distinguish the following cases:

$0 \le s \le v_\xi \le v$.
In this case for p large enough it holds

$$\begin{aligned} |G(v,s;p)| &\le \frac{2(S_c(\tau))^2}{m^*(p)S_c(v(\xi;p)) - S_c(\tau)} + S_c(\tau) \\ &\le \frac{2S_c\big(\int_0^1 \sqrt{b(s;p)}ds\big)}{m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac14} e^{\frac12\int_0^1 a(s)ds}\frac{S_c\big(\int_0^\xi \sqrt{b(s;p)}ds\big)}{S_c\big(\int_0^1 \sqrt{b(s;p)}ds\big)} - 1} \\ &\quad + S_c\Big(\int_0^1 \sqrt{b(s;p)}ds\Big). \end{aligned}$$

Thus due to Condition 10.1 there exists some $\hat{p}$ such that for all $p \ge \hat{p}$ it holds

$$|G(v,s;p)| \le \Big[\frac{2}{m(p)(\frac{\theta}{b_0})^{\frac14}\rho - 1} + 1\Big]k_1 \le 2k_1,$$

where

$$k_1 := \begin{cases} e^{\sqrt{b_0}}, & c = -1 \\ 1, & c = 1. \end{cases}$$

Also, we can easily see that, for large enough p the first partial derivative of G (with respect to v) satisfies

$$\begin{aligned}\Big|\frac{\partial}{\partial v}G(v,s;p)\Big| &\leq \frac{S_c(\tau)C_c(\tau)}{m^*(p)S_c(v(\xi;p))-S_c(\tau)}+C_c(\tau)\\ &\leq \frac{C_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}{m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}}e^{\frac{1}{2}\int_0^1 a(s)ds}\frac{S_c\big(\int_0^\xi\sqrt{b(s;p)}ds\big)}{S_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}-1}\\ &+C_c\Big(\int_0^1\sqrt{b(s;p)}ds\Big)\leq\frac{2k_1}{m(p)(\frac{\theta}{b_0})^{\frac{1}{4}}\rho-1}+2k_1\\ &=2k_1\Big[\frac{1}{m(p)(\frac{\theta}{b_0})^{\frac{1}{4}}\rho-1}+1\Big]\leq 4k_1.\end{aligned}$$

$0\leq v_\xi\leq s\leq v.$

In this case for p large enough it holds

$$\begin{aligned}|G(v,s;p)| &\leq \frac{(S_c(\tau))^2}{m^*(p)S_c(v(\xi;p))-S_c(\tau)}+S_c(\tau)\\ &\leq \frac{S_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}{m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}}e^{\frac{1}{2}\int_0^1 a(s)ds}\frac{S_c\big(\int_0^\xi\sqrt{b(s;p)}ds\big)}{S_c\big(\int_0^1\sqrt{b(s;p)}ds\big)}-1}\\ &+S_c\Big(\int_0^1\sqrt{b(s;p)}ds\Big)\leq\cdots\leq 2k_1.\end{aligned}$$

Similarly, we can obtain that for $0\leq v_\xi\leq s\leq v$ and p large enough, it holds

$$|G(v;s;p)|\leq 2k_1 \text{ and } \Big|\frac{\partial}{\partial v}G(v,s;p)\Big|\leq 4k_1,$$

while, for $0\leq v_\xi\leq v\leq s$, it holds

$$|G(v;s;p)|\leq k_1 \text{ and } \Big|\frac{\partial}{\partial v}G(v,s;p)\Big|\leq 2k_1.$$

Therefore for all s,v we have

$$\max\{|G(v,s;p)|,\ \Big|\frac{\partial}{\partial v}G(v,s;p)\Big|\}\leq 4k_1. \tag{10.6}$$

Applying the previous arguments we obtain that

$$|R(v;p)|\leq\frac{4k_1\rho_0 b_0^{\frac{1}{2}}}{\Delta_1}P(p)|x_0(p)|. \tag{10.7}$$

Here Δ is defined as

$$\Delta:=1-4k_1P(p)b_0^{\frac{1}{2}}=:\Delta_1(p)>0, \tag{10.8}$$

where $P(p)$ is defined in (3.15).

Hence the operator form of the boundary value problem (3.10)-(10.1) is the following:

$$\begin{aligned} y(v;p) &= w(v;p) \\ &+ \int_0^\tau G(v,s;p)C(\phi(s;p),y(s;p);p)y(s;p)ds, \quad v \in J_p. \end{aligned} \tag{10.9}$$

To show the existence of a solution of (10.9), as in Section 8, we consider the Banach space $C(J_p,\mathbb{R})$ of all continuous functions $y : J_p \to \mathbb{R}$ endowed with the sup-norm $\|\cdot\|$-topology. Fix a p large enough and define the operator $A : C(J_p,\mathbb{R}) \to C(J_p,\mathbb{R})$ by

$$(Az)(v) := w(v;p) + \int_0^\tau G(v,s;p)C(\phi(s;p),z(s);p)z(s)ds$$

which is completely continuous (due to Properties 3.1 and 3.3).

To proceed we assume for a moment that it holds

Take a large enough p and set $\tau = v(1;p) =: v$. Then we have $v \leq b_0^{\frac{1}{2}}$ and so it holds

$$1 - 4k_1P(p)\tau \geq \Delta_1 > 0.$$

Consider the open ball $B(0,l_1)$ in the space $C(J,\mathbb{R})$, where

$$l_1 := \frac{\|w\|}{1-4k_1P(p)\tau} + 1.$$

As in Section 8, assume that the operator A does not have any fixed point in $B(0,l_1)$. Thus, due to Theorem 8.2 and by setting $q = 0$, there exists a point z in the boundary of $B(0,l_1)$ satisfying

$$z = \lambda Az,$$

for some $\lambda \in (0,1)$. This means that for each $v \in J_p$ it holds

$$|z(v)| \leq \|w\| + \int_0^\tau |G(v,s;p)||C(\phi(s;p),z(s);p)||z(s)|ds.$$

Then we have

$$|z(v)| \leq \|w\| + 4k_1P(p)\int_0^\tau |z(s)|ds.$$

and therefore

$$|z(v)| \leq \|w\| + 4k_1P(p)\tau\|z\|,$$

which leads to the contradiction

$$l_1 = \|z\| \leq \frac{\|w\|}{1-4k_1P(p)\tau} = l_1 - 1.$$

Taking into account the relation between the solutions of the original problem and the solution of the problem (1.9)-(1.8), as well the previous arguments, we conclude the following result:

Theorem 10.2. *If Properties 3.1, 3.3 and* (10.8) *are true, then the boundary value problem* (1.9)-(1.8) *admits at least one solution.*

Now, we give the main results of this section. If w is the function defined in (10.2) we define the function

$$\begin{aligned}\tilde{x}(t;p) :&= Y(t;p)w(v(t;p);p)\\ &= Y(t;p)\frac{S_c(\tau - v) - m^* S_c(v(\xi;p) - v)}{S_c(\tau) - m^* S_c(v(\xi;p))}y_0(p)\\ &= \Big(\frac{b(0;p)}{b(t;p)}\Big)^{\frac{1}{4}} \exp\Big(-\frac{1}{2}\int_0^t a(s;p)ds\Big)\frac{X(t;p)}{X(0;p)}x_0(p),\end{aligned} \tag{10.10}$$

where

$$\begin{aligned}X(t;p) :&= S_c\Big(\int_t^1 \sqrt{b(s;p)}ds\Big)\\ &\quad - m(p)\Big(\frac{b(1;p)}{b(\xi;p)}\Big)^{\frac{1}{4}} e^{\frac{1}{2}\int_\xi^1 a(s)ds} S_c\Big(\int_t^\xi \sqrt{b(s;p)}ds\Big),\end{aligned}$$

which, as we shall show, it is an approximate solution of the problem under discussion.

Theorem 10.3. *Consider the boundary value problem* (1.9)-(1.8), *where assume that Properties 3.1, 3.3, 8.1 , the conditions* (10.8) *and (i), (ii) of Theorem 4.2 keep in force. Also, assume that* $x_0 \in \mathcal{A}_E$.

a) If the condition

$$\min_{j=1}^{5} \mathcal{G}_E(\Phi_j) + \mathcal{G}_E(x_0) =: L > 0 \tag{10.11}$$

is satisfied, then the existence of a solution x *of the problem is guaranteed and if*

$$\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p)$$

is the error function, where $\tilde{x}$ *is defined by* (8.14), *then we have*

$$\mathcal{E}(t;p) \simeq 0, \ \ p \to +\infty, \ \ t \in Co([0,1]). \tag{10.12}$$

Also, the growth index at infinity of the error function satisfies

$$\mathcal{G}_E(\mathcal{E}(t;\cdot)) \geq L, \ \ t \in Co([0,1]). \tag{10.13}$$

b) Moreover we have

$$\frac{d}{dt}\mathcal{E}(t;p) \simeq 0, \ \ p \to +\infty, \ \ t \in Co([0,1]), \tag{10.14}$$

and

$$\mathcal{G}_E(\frac{d}{dt}\mathcal{E}(t;\cdot)) \geq L, \ \ t \in Co([0,1]). \tag{10.15}$$

Proof. a) Take a $N \in (0, L)$ and choose $\zeta > 0$ as well as $-\sigma < \mathcal{G}_E(x_0)$, (thus we have

$$|x_0(p)| \leq K_3(E(p))^{\sigma},$$

for some $K_3 > 0$) such that

$$\min_{j=1}^{5} \lambda(\Phi_j) > \zeta \geq N + \sigma. \tag{10.16}$$

Therefore it follows that

$$\sigma - \zeta \leq -N \tag{10.17}$$

and

$$P(p) \leq K(E(p))^{-\zeta}, \tag{10.18}$$

for some $K > 0$. Thus (10.8) keeps in force for p large enough. This makes Theorem 10.2 applicable and the existence of a solution is guaranteed.

Let $\mathcal{E}(t;p)$ be the error function defined in (8.17). From (10.7) it is easy to obtain that

$$|\mathcal{E}(t;p)| \leq \Lambda_1(E(p))^{\sigma-\zeta}.$$

for all large p, for some $\Lambda_1 > 0$. Obviously, this relation implies (10.12) as well as (10.13).

b) Next consider the first order derivative of the error function $\mathcal{E}(t;p)$. Again, as above, we obtain

$$\begin{aligned}|\frac{d}{dt}R(v(t;p);p)| &= |\frac{d}{dv}\int_0^{\tau} G(v,s;p)H(s;p)ds\frac{d}{t}v(t;p)| \\ &\leq Y(t;p)\Big[\frac{1}{4}\sqrt{\Phi_1(p)b(0;p)} + \frac{a(t;p)}{2} \\ &+ \int_0^{\tau}\Big(G(v,s;p)| + |\frac{d}{dt}v(t;p)||\frac{\partial}{\partial v}G(v,s;p)|\Big)|H(s;p)|ds\Big].\end{aligned}$$

Now, we use (10.16), (10.18), (10.17), (10.6), (10.5) and (10.7) to conclude that for some positive constants k_3, k_4 it holds

$$|\frac{d}{dt}\mathcal{E}(t;p)| \leq k_3 P(p)|x_0(p)| \leq k_4(E(p))^{\sigma-\zeta} < k_4(E(p))^{-N},$$

from which the result follows. □

11. An application

Consider the equation

$$x'' + x' + x + \frac{x\sin(x)}{p} = 0, \quad t \in [0,1] \tag{11.1}$$

associated with the following boundary value conditions:

$$x(0;p) = p^{-1}, \quad x(1;p) = e^p x(\frac{1}{2};p). \tag{11.2}$$

We can easily see that with respect to the unbounded function $E(p) :=$ p we have

$$\mathcal{G}_E(\Phi_j) = 1, \quad j = 1,2,3,4,5 \text{ and } \mathcal{G}_E(x_0) = 2.$$

Therefore $L = 2$ and, so, Theorem 10.3 applies. This means that there is a solution of the problem (11.1)-(11.2) and an approximate solution of it is the following (according to (10.10)):

$$\tilde{x}(t;p) := \frac{\sin(1-t) - e^p e^{\frac{1}{4}} \sin(\frac{1}{2} - t)}{\sin(1) - e^p e^{\frac{1}{4}} \sin(\frac{1}{2})} e^{-\frac{t}{2}} p^{-2}, \quad t \in [0,1].$$

The graph of this function for the values of $p = 3.83, 6.33, 8.83, 15.50$ is shown in Figure 5

12. Discussion

We have presented a method of computing the approximate solutions of two initial value problems and two boundary value problems concerning the second order ordinary differential equation (1.5). First of all in section 2 we have given the meaning of measuring the approximation, by introducing the growth index of a function. It is proved that this meaning helps a lot to get information on how close to the actual solution is the approximate solution as the parameter p tends to $+\infty$. Section 3 of the work provided the first step of the method, since therein we have shown the way of transforming by (3.1) the original equation to an auxiliary and easy to elaborate differential equation (3.10).

The sign of the response coefficient $b(t;p)$ plays an essential role. If it is positive, we have an wave featured solution, while in case it is negative we have exponential picture. This is the reason for discussing the two cases separately especially in the initial value problems. The first case is exhibited in Section 4, where in Theorem 4.1 we show first the existence of a solution of the initial value problem and prepare the ground for the existence of C^1-approximate solutions provided in Theorems 4.2 and Theorem 4.3. The two theorems give, mainly, similar

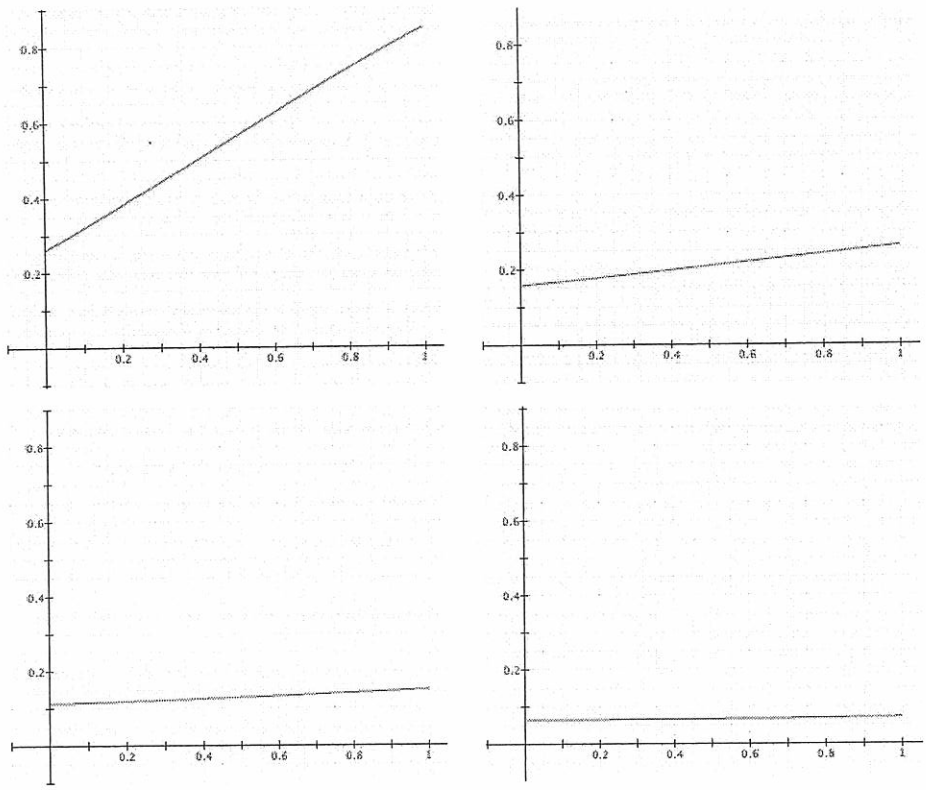

FIGURE 5. Approximate solutions of (11.1) - (11.2), when p=3.83, 6.33, 8.83, 15.50, respectively.

results, but in the first theorem we assumed that the coefficient $a(t;p)$ is positive and in the second it is assumed that it may take negative values as well.

An application of the results in an example where the two coefficients $a(t;p)$ and $b(t;p)$ are positive, is given in Section 5, where the C^1-approximate solution is computed. The case of negative $b(t;p)$ is discussed in section 6 and the approximate results are applied to a initial value problem in Section 7.

The boundary value problem (1.9)-(1.10) is discussed in Section 8. First by the help of the (Fixed Point Theorem of) Nonlinear Alternative we have guaranteed in Theorem 8.3 the existence of solutions of the problem. Then, in Theorem 8.4 we gave estimates of the error function $\mathcal{E}(t;p) := x(t;p) - \tilde{x}(t;p)$, where $\tilde{x}(t;p)$ is the C^1-approximate solution. Here we are able to give simultaneously our results in the cases of positive and negative $b(t;p)$. A specific case when $a(t;p)$ is nonnegative and the solution vanishes in an edge of the existence interval is discussed separately in Theorem 8.5, while two applications of the results where given in Section 9.

In Section 10 we investigated the boundary value problem (1.9)-(1.8). Again, first in Theorem 10.2 we solved the existence problem by using the Nonlinear Alternative and then we proceeded to the proof of the existence of C^1-approximate solutions in Theorem 10.3. An application to specific equation is given in the last section 11.

Notice that all examples which we have presented are associated with some pictures[2], which show the change of the approximate solutions, as the parameter p takes large values and tends to $+\infty$.

As we have seen, in order to apply the method to a problem we have to do two things: First to transform the original equation to a new one and then to transform the initial values or the boundary values to the new ones. Both of them are important in the process of the method.

And as the transformation of the original equation was already given in (3.10), what one has to do is to proceed to the transformation of the boundary values. For instance, in case the boundary values of the original problem are of the form

$$x(0;p) = x'(0;p), \quad x(1;p) = x'(1;p),$$

then, it is not hard to show that, under the transformation S_p the new function $y(\cdot;p)$ is required to satisfy the boundary values

$$y'(0;p) = \frac{1}{\sqrt{b(0;p)}}\Big[1 + \frac{1}{4}\frac{b'(0;p)}{b(0;p)} + \frac{1}{2}a(0;p)\Big]y(0;p)$$

and

$$y'(\tau;p) = \frac{1}{\sqrt{b(1;p)}}\Big[1 + \frac{1}{4}\frac{b'(1;p)}{b(1;p)} + \frac{1}{2}a(1;p)\Big]y(1;p).$$

Now one can proceed to the investigation of the existence of approximate solutions as well as to their computation.

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